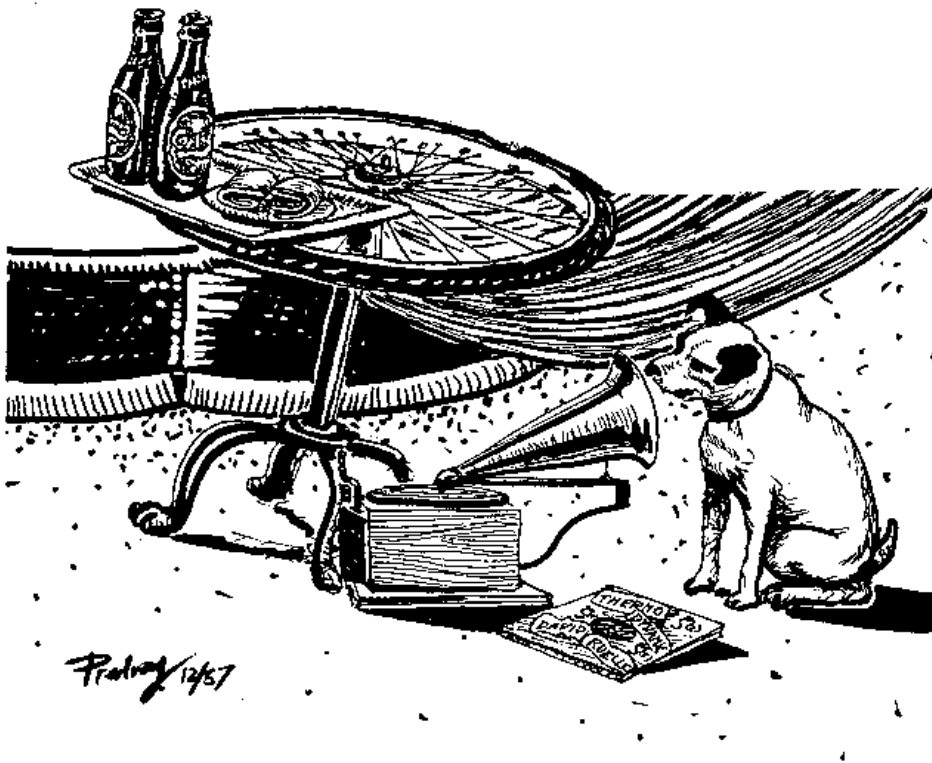


Chaos: Classical and Quantum

I: Deterministic Chaos



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Gábor Vattay

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Contributors

No man but a blockhead ever wrote except for money
—Samuel Johnson

This book is a result of collaborative labors of many people over a span of several decades. Coauthors of a chapter or a section are indicated in the byline to the chapter/section title. If you are referring to a specific coauthored section rather than the entire book, cite it as (for example):

C. Chandre, F.K. Diakonov and P. Schmelcher, section “Discrete cyclist relaxation method,” in P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner and G. Vattay, *Chaos: Classical and Quantum* (Niels Bohr Institute, Copenhagen 2010); ChaosBook.org/version13.

Do not cite chapters by their numbers, as those change from version to version. Chapters without a byline are written by Predrag Cvitanović. Friends whose contributions and ideas were invaluable to us but have not contributed written text to this book, are credited in the acknowledgments.

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Daniel Borrero Oct 23 2008, `soluCycles.tex`

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Lei Zhang

Solutions 1.1, 2.1

Acknowledgments

I feel I never want to write another book. What's the good!
I can eke living on stories and little articles, that don't cost
a tittle of the output a book costs. Why write novels any
more!

—D.H. Lawrence

This book owes its existence to the Niels Bohr Institute's and Nordita's hospitable and nurturing environment, and the private, national and cross-national foundations that have supported the collaborators' research over a span of several decades. P.C. thanks M.J. Feigenbaum of Rockefeller University; D. Ruelle of I.H.E.S., Bures-sur-Yvette; I. Procaccia of the Weizmann Institute; P.H. Damgaard of the Niels Bohr International Academy; G. Mazenko of U. of Chicago James Franck Institute and Argonne National Laboratory; T. Geisel of Max-Planck-Institut für Dynamik und Selbstorganisation, Göttingen; I. Andrić of Rudjer Bošković Institute; P. Hemmer of University of Trondheim; The Max-Planck Institut für Mathematik, Bonn; J. Lowenstein of New York University; Edificio Celi, Milano; Fundação de Faca, Porto Seguro; and Dr. Dj. Cvitanović, Kostrena, for the hospitality during various stages of this work, and the Carlsberg Foundation, Glen P. Robinson, Humboldt Foundation and National Science Foundation grant DMS-0807574 for partial support.

The authors gratefully acknowledge collaborations and/or stimulating discussions with E. Aurell, M. Avila, V. Baladi, D. Barkley, B. Brenner, A. de Carvalho, D.J. Driebe, B. Eckhardt, M.J. Feigenbaum, J. Frøjlund, S. Froehlich, P. Gaspar, P. Gaspard, J. Guckenheimer, G.H. Gunaratne, P. Grassberger, H. Gutowitz, M. Gutzwiller, K.T. Hansen, P.J. Holmes, T. Janssen, R. Klages, Y. Lan, B. Lautritzen, J. Milnor, M. Nordahl, I. Procaccia, J.M. Robbins, P.E. Rosenqvist, D. Ruelle, G. Russberg, B. Sandstede, M. Sieber, D. Sullivan, N. Søndergaard, T. Tél, C. Tresser, R. Wilczak, and D. Wintgen.

We thank Dorte Glass, Tzatzilha Torres Guadarrama and Raenell Soller for typing parts of the manuscript; D. Borrero, B. Lautrup, J.F Gibson and D. Viswanath for comments and corrections to the preliminary versions of this text; M.A. Porter for the quotation on page 138 and for lengthening the manuscript by the 2013 definite articles hitherto missing; M.V. Berry for the quotation on page 747; H. Fogedby for the quotation on page 456; J. Greensite for the quotation on page 7; Ya.B. Pesin for the remarks quoted on page 762; M.A. Porter for the quotations on page 21 and page 754; E.A. Spiegel for quotation on page 3; and E. Valesco for the quotation on page 25.

F. Haake's heartfelt lament on page 370 was uttered at the end of the first conference presentation of cycle expansions, in 1988. G.P. Morriss advice to students as how to read the introduction to this book, page 6, was offered during a 2002 graduate course in Dresden. K. Huang's C.N. Yang interview quoted on page 332 is available on ChaosBook.org/extras. T.D. Lee remarks on as to who is to blame, page 37 and page 265, as well as M. Shub's helpful technical remark on page 468 came during the Rockefeller University December 2004 "Feigenbaum Fest." Quotes on pages 37, 126, and 329 are taken from a book review by J. Guckenheimer [1.1].

Who is the 3-legged dog reappearing throughout the book? Long ago, when

we were innocent and knew not Borel measurable α to Ω sets, P. Cvitanović asked V. Baladi a question about dynamical zeta functions, who then asked J.-P. Eckmann, who then asked D. Ruelle. The answer was transmitted back: “The master says: ‘It is holomorphic in a strip’.” Hence His Master’s Voice logo, and the 3-legged dog is us, still eager to fetch the bone. The answer has made it to the book, though not precisely in His Master’s voice. As a matter of fact, the answer *is* the book. We are still chewing on it.

Profound thanks to all the unsung heroes—students and colleagues, too numerous to list here—who have supported this project over many years in many ways, by surviving pilot courses based on this book, by providing invaluable insights, by teaching us, by inspiring us.