Calculating the Dynamical Zeta Function and the Fredholm Determinant Using Mathematica

The programs *orbits.m zeta.m* and *fredholm.m* calculate the zeta function and fredholm determinant for the pinball. To run the programs copy the files into a directory and start up mathematica.

The DynamicalZeta Function

To calculate the dynamical zeta function load the file *zeta.m* using

```
In[1]:= << zeta.m
```

This reads in *zeta.m*. The first command in the file is

```
Get["orbit.m"]
```

which is equivalent to "<< orbit.m" and reads in orbit.m. This consists of a list of periodic orbits giving their length, np[i], period, Tp[i], and stability, lam[i].

```
np[0] = 1; Tp[0] =   4.0000000000000000; lam[0] =   9.8989794855663575
np[1] = 1; Tp[1] =   4.2679491924311233; lam[1] = -11.7714551963855620
np[2] = 2; Tp[2] =   8.3165294851683011; lam[2] = -124.0948019921975500
```

etc. The file contains all 40 orbits of length 7 or less. The next line in *zeta.m* defines $t_p(s,z)$

$$t_p(s,z) = \frac{\mathrm{e}^{sT_p} z_p^n}{|\Lambda_p|} \tag{1}$$

```
t[n_] := Exp[s Tp[n]] z^np[n]/Abs[lam[n]]
```

The next command defines the zeta function

$$\frac{1}{\zeta(s,z)} = \prod_p (1 - t_p) = \sum b_n z^n \tag{2}$$

```
zeta = Series[Product[(1-t[i]), {i,0,40}], {z,0,7}]
```

Here we get mathematica to expand out the (now finite) product and then to re-sum the series in terms of a taylor series. Since we have neglected cycles in the expansion of length 8 and greater this expansion is only correct up to 7^{th} order. We next set $z = 1$ and find the leading zero of the resulting expression. To see how the expression converges we define a function gamma[n] which gives the zero keeping only terms up to order z^n

```
gamma[m_] := Block[{zeta1},
  zeta1 = Normal[Series[zeta, {z,0,m}]] /. z->1;
  N[s /. FindRoot[zeta1==0, {s, 0.4}], 16]
]
```

The function "Normal" converts the taylor expansion back into a polynomial while the expression "/. z$->$1" sets "z" to 1. "FindRoot" uses Newton-Raphson to find the root of the equation starting from an initial guess of 0.4. We next compute the 7 estimates to γ using

```
gam = Table[gamma[i], {i,1,7}]
```

This is printed out using

```
Print["Escape rate = ", gam]
```

The obtain the following estimates for γ 0.4076937103368145, 0.4102804728701962, 0.41033-67116692595, 0.4103383541007433, 0.4103384061041494, 0.4103384077187464, 0.41033840-77678522. Comparing with the answer in the notes the final estimate is accurate up to 11 decimal places. To study the convergence further we can look at the function $1/\zeta(s = s_0, z)$ as a function of z. We first substitute in our best estimate for $s_0 = \gamma$

```
zeta1 = zeta /. s -> gamma[7]
```

then we can plot the function by

```
Plot[Evaluate[Normal[zeta1]], {z, -100, 100}]
```

or plot the coefficients b_n using

```
ListPlot[Log[Abs[CoefficientList[zeta1]].
```

The Fredholm Determinant

To calculate the Fredholm determinant load *fredholm.m* using

```
In[1]:= << fredholm.m
```

The first two commands are the same as for the dynamical zeta function. The next command defines

$$B_p(r) = \frac{-t_p^r}{r(1-\Lambda^{-r})^2} \tag{3}$$

```
B[p_,r_] := -t[p]^r/(r (1-lam[p]^(-r))^2)
```

The logarithm of the Fredholm determinant is derined by

$$\log(F(s,z)) = \sum_p \sum_{r=1} B_p(r) \tag{4}$$

we calculate this up to 7^{th} order in z using

```
logFred = Series[Sum[Sum[B[p,r] , {r,1,Floor[7/np[p]]}], {p,0,40}], {z,0,7}]
```

We then take the exponential to find $F(s, z)$

```
Fred = Exp[logFred]
```

Mathematica is sufficiently clever to know how to take functions of series, however, this operation might involve subtracting large numbers of nearly equal size which can lead to a loss of accuracy. By default mathematica works to a set precision equal to "$Precision", on my machine this is 16 decimal places. Mathematica can work at any precision but as the periodic orbits are only given to double precision it is not worth making mathematica work more accurately.

The rest of fredholm.m is essentially identical to zeta.m. Because of the inaccuracy of the orbits the estimate for γ does not improve using more than 5 terms. The estimates are 0.3958876339245904, 0.4105693452542427, 0.4103381379516226, 0.4103384074110521, 0.4103384077696336, 0.4103384077693569 and 0.4103384077693568.

The logarithm of the coefficients for the zeta function and Fredholm determinants are shown in figure1

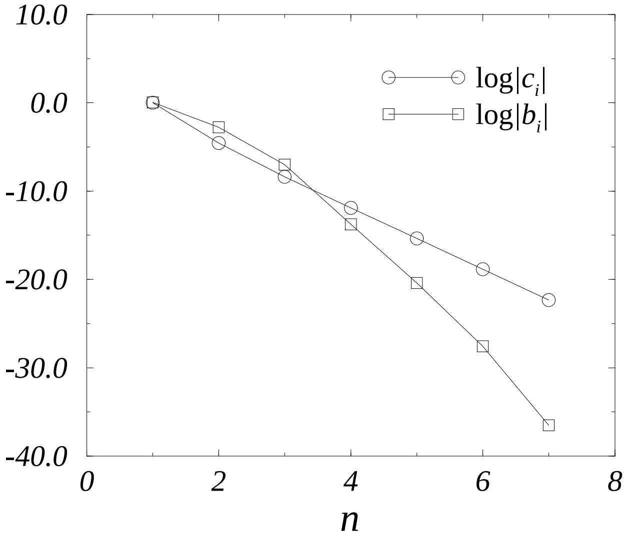

Figure 1: Plot of the logarithms of the coefficients for the dynamical zeta function, b_n and the Fredholm determinant c_n. Note that the coefficients for the Fredholm determinant may be inaccurate for large n due to rounding errors.