

dynamical zeta functions: what, why and what are the good for?

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I accept chaos

I am not sure that it accepts me

—Bob Dylan, *Bringing It All Back Home*

in physics, no problem is tractable

requires summing up exponentially increasing # of
exponentially decreasing terms

yet

in practice

every physical problem must be tractable

“can't do” doesn't cut it

dynamical systems

state space

a manifold $\mathcal{M} \in \mathbb{R}^d$: d numbers determine the state of the system

representative point

$$x(t) \in \mathcal{M}$$

a state of physical system at instant in time

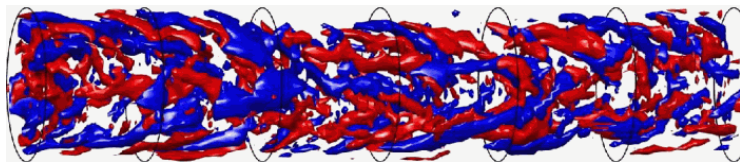
today's experiments

example of a representative point

$$x(t) \in \mathcal{M}, d = \infty$$

a state of turbulent pipe flow at instant in time

Stereoscopic Particle Image Velocimetry \rightarrow 3-*d* velocity field over the entire pipe¹

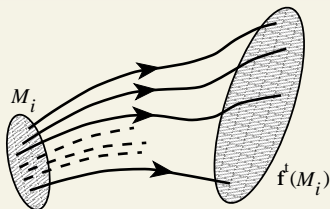


¹Casimir W.H. van Doorne (PhD thesis, Delft 2004)

dynamics

map $f^t(x_0)$ = representative point time t later

evolution



f^t maps a region \mathcal{M}_i of the state space into the region $f^t(\mathcal{M}_i)$.

dynamics defined

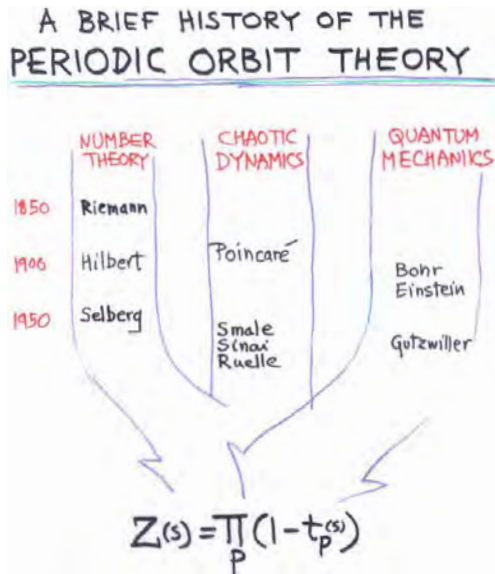
dynamical system

the pair (\mathcal{M}, f)

the problem

enumerate, classify all solutions of (\mathcal{M}, f)

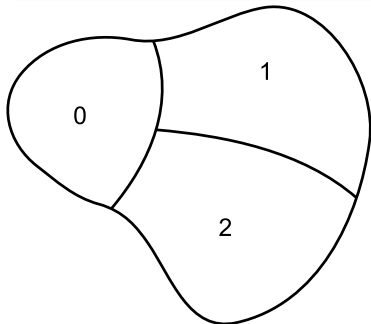
one needs to enumerate \rightarrow hence **zeta functions** !



state space, partitioned

partition into regions of similar states

state space coarse partition

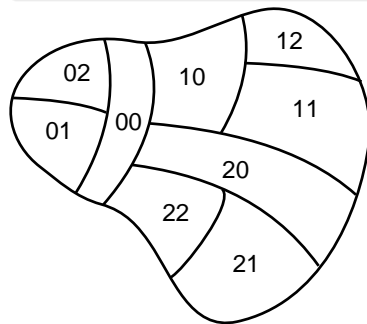


$$\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2$$

ternary alphabet

$$\mathcal{A} = \{1, 2, 3\}.$$

1-step memory refinement



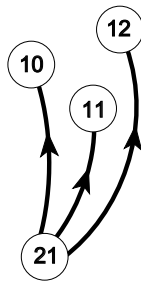
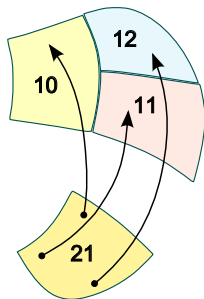
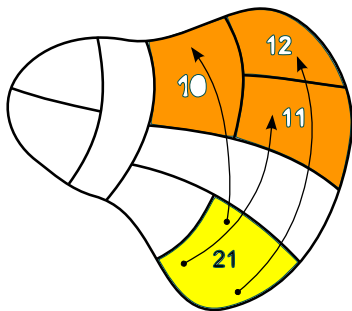
$$\mathcal{M}_i = \mathcal{M}_{i0} \cup \mathcal{M}_{i1} \cup \mathcal{M}_{i2}$$

labeled by nine 'words'

$$\{00, 01, 02, \dots, 21, 22\}.$$

state space, partitioned

topological dynamics



one time step

points from \mathcal{M}_{21}
reach $\{\mathcal{M}_{10}, \mathcal{M}_{11}, \mathcal{M}_{12}\}$
and no other regions

each region = node

allowed transitions

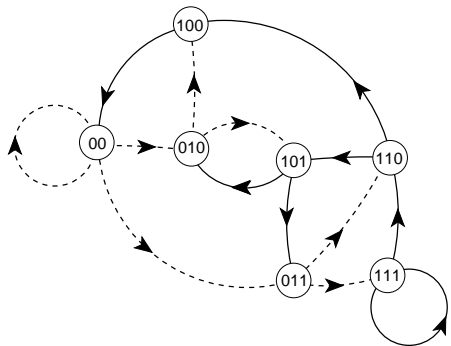
$$T_{10,21} = T_{11,21} = T_{12,21} \neq 0$$

directed links

state space, partitioned

topological dynamics

Transition graph T_{ba}
 regions reached in one
 time step



example: state space resolved into 7 neighborhoods

$$\{\mathcal{M}_{00}, \mathcal{M}_{011}, \mathcal{M}_{010}, \mathcal{M}_{110}, \mathcal{M}_{111}, \mathcal{M}_{101}, \mathcal{M}_{100}\}$$

how many ways to get there from here?

$$\begin{aligned}(T^n)_{ij} &= \sum_{k_1, k_2, \dots, k_{n-1}} T_{ik_1} T_{k_1 k_2} \cdots T_{k_{n-1} j} \\ &\propto \lambda_0^n, \quad \lambda_0 = \text{leading eigenvalue}\end{aligned}$$

counts topologically distinct n -step paths
starting in \mathcal{M}_j and ending in partition \mathcal{M}_i .

compute eigenvalues by evaluating the determinant

topological (or Artin-Mazur) zeta function

$$1/\zeta_{\text{top}} = \det(1 - zT)$$

topological dynamics

$$\det(1 - zT) = \sum [\text{non-(self)-intersecting loops}]$$

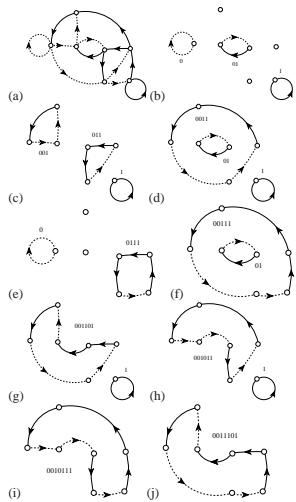


Figure 15.1: (a) The region labels in the nodes of transition graph figure 14.3 can be omitted, as the links alone keep track of the symbolic dynamics. (b)-(j) The fundamental cycles (15.23) for the transition graph (a), i.e., the set of its non-self-intersecting loops. Each loop represents a local trace t_p , as in (14.5).

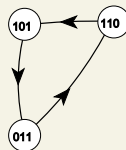
Example 15.5 Nontrivial pruning: The non-self-intersecting loops of the transition graph of figure 14.6(d) are indicated in figure 14.6(e). The determinant can be written down by inspection, as the sum of all possible partitions of the graph into products of

periodic orbit

loop, periodic orbit, cycle

walk that ends at the starting node, for example

$$t_{011} = L_{110,011} L_{011,101} L_{101,110} =$$



zeta function (“partition function”)

$$\det(1 - zT)$$

can be read off the graph, expanded as a polynomial in z , with coefficients given by products of non-intersecting loops (traces of powers of T)

cycle expansion of a zeta function

$$\begin{aligned} \det(1 - zT) = & 1 - (t_0 + t_1)z - (t_{01} - t_0 t_1) z^2 \\ & - (t_{001} + t_{011} - t_{01} t_0 - t_{01} t_1) z^3 \\ & - (t_{0011} + t_{0111} - t_{001} t_1 - t_{011} t_0 - t_{011} t_1 + t_{01} t_0 t_1) z^4 \\ & - (t_{00111} - t_{0111} t_0 - t_{0011} t_1 + t_{011} t_0 t_1) z^5 \\ & - (t_{001011} + t_{001101} - t_{0011} t_{01} - t_{001} t_{011}) z^6 \\ & - (t_{0010111} + t_{0011101} - t_{001011} t_1 - t_{001101} t_1 \\ & \quad - t_{00111} t_{01} + t_{0011} t_{01} t_1 + t_{001} t_{011} t_1) z^7 \end{aligned}$$

if there is one idea that one should learn about chaotic dynamics

it is this

there is a fundamental local \leftrightarrow global duality which says that

eigenvalue spectrum is dual to periodic orbits spectrum

for dynamics on the circle, this is called Fourier analysis

for dynamics on well-tiled manifolds, Selberg traces and zetas

for generic nonlinear dynamical systems the duality is embodied in the trace formulas and zeta functions

global eigenspectrum \Leftrightarrow local periodic orbits

Twenty years of schooling
and they put you on the day shift
Look out kid, they keep it all hid

—Bob Dylan, *Subterranean Homesick Blues*

the eigenspectrum s_0, s_1, \dots of the classical evolution operator

trace formula, infinitely fine partition

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_{\alpha}} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta \cdot A_p - s T_p)}}{|\det(\mathbf{1} - M_p^r)|}.$$

the beauty of trace formulas lies in the fact that everything on the right-hand-side

–prime cycles p , their periods T_p and the eigenvalues of M_p –

is a coordinate independent, invariant property of the flow

deterministic chaos vs. noise

any physical system:

noise limits the resolution that can be attained in partitioning the state space

noisy orbits

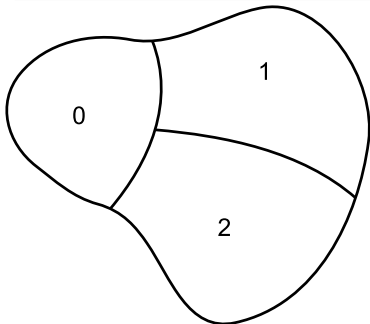
probabilistic densities smeared out by the noise:
a finite # fits into the attractor

goal: determine

the **finest attainable** partition

deterministic partition

state space coarse partition

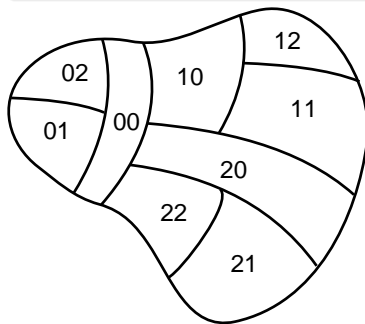


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1-step memory refinement

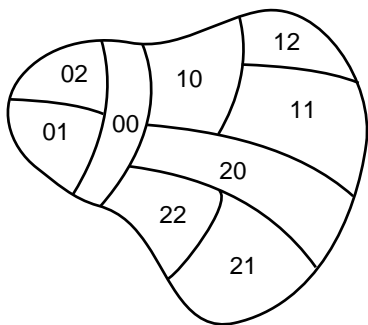


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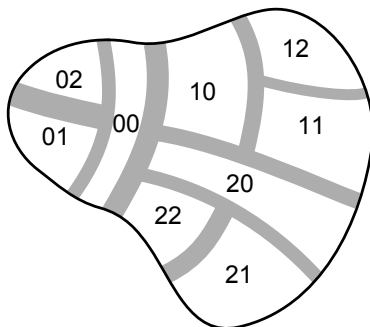
$$\{00, 01, 02, \dots, 21, 22\}.$$

deterministic vs. noisy partitions



deterministic partition

can be refined
ad infinitum



noise blurs the boundaries

when overlapping, no further
refinement of partition

idea #1: partition by periodic points

periodic points instead of boundaries

- each partition contains a short periodic point smeared into a 'cigar' by noise

idea #1: partition by periodic points

periodic points instead of boundaries

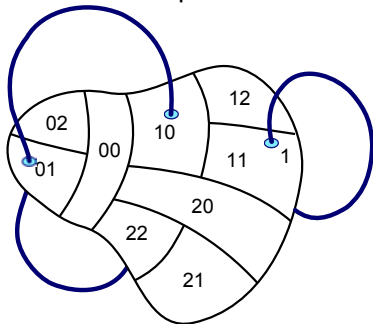
- each partition contains a short periodic point smeared into a 'cigar' by noise

compute the size of a noisy periodic point neighborhood

idea #1: partition by periodic points

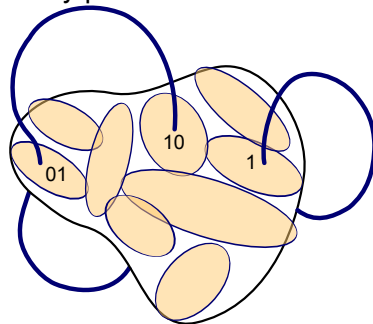
periodic orbit partition

deterministic partition



some short periodic points:
fixed point $\bar{1} = \{x_1\}$
two-cycle $\overline{01} = \{x_{01}, x_{10}\}$

noisy partition



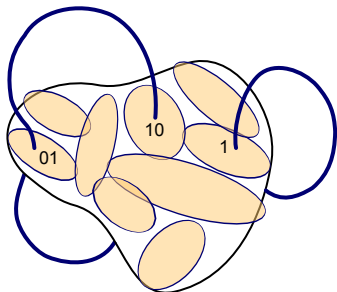
periodic points blurred by the
Langevin noise into
cigar-shaped densities

- successive refinements of a deterministic partition: exponentially shrinking neighborhoods
- as the periods of periodic orbits increase, the diffusion always wins:

partition stops at the finest attainable partition, beyond which the diffusive smearing exceeds the size of any deterministic subpartition.

idea #1: partition by periodic points

noisy periodic orbit partition



optimal partition hypothesis

optimal partition:
the maximal set of resolvable
periodic point neighborhoods

why care?

if the high-dimensional flow has only a few unstable directions, the overlapping stochastic 'cigars' provide a *compact cover* of the noisy chaotic attractor, embedded in a state space of arbitrarily high dimension

strategy

- use periodic orbits to partition state space
- compute local eigenfunctions of the Fokker-Planck operator to determine their neighborhoods
- done once neighborhoods overlap

idea #2: evolve densities, not Langevin trajectories

how big is the neighborhood blurred by the Langevin noise?

the (well known) key formula

composition law for the covariance matrix Q_a

$$Q_{a+1} = M_a Q_a M_a^T + \Delta_a$$

density covariance matrix at time a : Q_a

Langevin noise covariance matrix: Δ_a

Jacobian matrix of linearized flow: M_a

idea #2: evolve densities, not Langevin trajectories

roll your own cigar

evolution law for the covariance matrix Q_a

$$Q_{a+1} = M_a Q_a M_a^T + \Delta_a$$

in one time step a Gaussian density distribution with covariance matrix Q_a is smeared into a Gaussian 'cigar' whose widths and orientation are given by eigenvalues and eigenvectors of Q_{a+1}

- (1) deterministically advected and deformed
local density covariance matrix $Q \rightarrow MQM^T$
- (2) add noise covariance matrix Δ

add up as sums of squares

idea #2: evolve densities, not Langevin trajectories

noise along a trajectory

iterate $Q_{a+1} = M_a Q_a M_a^T + \Delta_a$ along the trajectory

if M is contracting, over time the memory of the covariance Q_{a-n} of the starting density is lost, with iteration leading to the limit distribution

$$Q_a = \Delta_a + M_{a-1} \Delta_{a-1} M_{a-1}^T + M_{a-2}^2 \Delta_{a-2} (M_{a-2}^2)^T + \dots$$

diffusive dynamics of a nonlinear system is fundamentally different from Brownian motion, as the flow induces a history dependent effective noise:

Always!

idea #2: evolve densities, not Langevin trajectories

noise and a single attractive fixed point

if all eigenvalues of M are strictly contracting, any initial compact measure converges to the unique invariant Gaussian measure $\rho_0(z)$ whose covariance matrix satisfies

time-invariant measure condition (Lyapunov equation)

$$Q = MQM^T + \Delta$$

[A. M. Lyapunov 1892, doctoral dissertation]

idea #2: evolve densities, not Langevin trajectories

example : Ornstein-Uhlenbeck process

width of the natural measure concentrated at the deterministic fixed point $z = 0$

$$Q = \frac{2D}{1 - |\Lambda|^2}, \quad \rho_0(z) = \frac{1}{\sqrt{2\pi Q}} \exp\left(-\frac{z^2}{2Q}\right),$$

- is balance between contraction by Λ and diffusive smearing by $2D$ at each time step
- for strongly contracting Λ , the width is due to the noise only
- As $|\Lambda| \rightarrow 1$ the width diverges: the trajectories are no longer confined, but diffuse by Brownian motion

idea #3: for unstable directions, look back

things fall apart, centre cannot hold

but what if M has *expanding* Floquet multipliers?

both deterministic dynamics and noise tend to smear densities away from the fixed point: no peaked Gaussian in your future

idea #3: for unstable directions, look back

things fall apart, centre cannot hold

but what if M has *expanding* Floquet multipliers?

Fokker-Planck operator is non-selfadjoint

If right eigenvector is peaked (attracting fixed point)
the left eigenvector is flat (probability conservation)

idea #3: for unstable directions, look back

case of *repelling* fixed point

if M has only *expanding* Floquet multipliers, both deterministic dynamics and noise tend to smear densities away from the fixed point

balance between the two is described by the *adjoint Fokker-Planck operator*, and the evolution of the covariance matrix Q is now given by

$$Q_a + \Delta = M_a Q_{a+1} M_a^T,$$

[aside to control freaks: reachability and observability Gramians]

optimal partition challenge

finally in position to address our challenge:

determine the finest possible partition for a given noise

resolution of a one-dimensional chaotic repeller

As an illustration of the method, consider the chaotic repeller on the unit interval

$$x_{n+1} = \Lambda_0 x_n(1 - x_n)(1 - bx_n) + \xi_n, \quad \Lambda_0 = 8, \quad b = 0.6,$$

with noise strength $2D = 0.002$

'the best possible of all partitions' hypothesis formulated as an algorithm

- calculate the local adjoint Fokker-Planck operator eigenfunction width Q_a for every unstable periodic point x_a
- assign one-standard deviation neighborhood $[x_a - Q_a, x_a + Q_a]$ to every unstable periodic point x_a
- cover the state space with neighborhoods of orbit points of higher and higher period n_p
- stop refining the local resolution whenever the adjacent neighborhoods of x_a and x_b overlap:

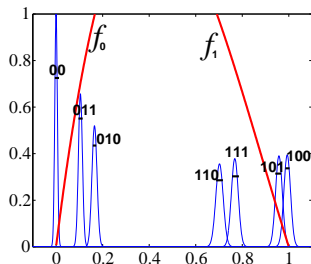
$$|x_a - x_b| < Q_a + Q_b$$

optimal partition, 1 dimensional map

f_0, f_1 : branches of deterministic map

local eigenfunctions $\tilde{\rho}_a$ partition state space by neighborhoods of periodic points of period 3

neighborhoods \mathcal{M}_{000} and \mathcal{M}_{001} overlap, so \mathcal{M}_{00} cannot be resolved further



all neighborhoods $\{\mathcal{M}_{0101}, \mathcal{M}_{0100}, \dots\}$ of period $n_p = 4$ cycle points overlap, so

state space can be resolved into 7 neighborhoods

$$\{\mathcal{M}_{00}, \mathcal{M}_{011}, \mathcal{M}_{010}, \mathcal{M}_{110}, \mathcal{M}_{111}, \mathcal{M}_{101}, \mathcal{M}_{100}\}$$

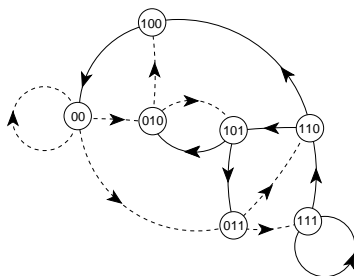
Markov partition

evolution in time maps intervals

$$\mathcal{M}_{011} \rightarrow \{\mathcal{M}_{110}, \mathcal{M}_{111}\}$$

$$\mathcal{M}_{00} \rightarrow \{\mathcal{M}_{00}, \mathcal{M}_{011}, \mathcal{M}_{010}\}, \text{ etc..}$$

summarized by the transition graph (links correspond to elements of transition matrix T_{ba}):
the regions b that can be reached from the region a in one time step



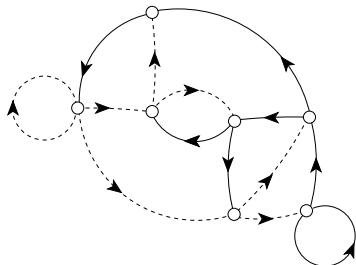
transition graph

7 nodes = 7 regions of the optimal partition

dotted links = symbol 0 (next region reached by f_0)

full links = symbol 1 (next region reached by f_1)

region labels in the nodes can be omitted, with links keeping track of the symbolic dynamics



- (1) deterministic dynamics is full binary shift, but
- (2) noise dynamics nontrivial and *finite*

predictions

escape rate and the Lyapunov exponent of the repeller

are given by the leading eigenvalue of this $[7 \times 7]$ graph / transition matrix

tests : numerical results are consistent with the full Fokker-Planck PDE simulations

what is novel?

- we have shown how to compute the **locally optimal partition**, for a given dynamical system and given noise, in terms of local eigenfunctions of the forward-backward actions of the Fokker-Planck operator and its adjoint

what is novel?

- **A handsome reward:** as the optimal partition is always finite, the dynamics on this 'best possible of all partitions' is encoded by a finite transition graph of finite memory, and the Fokker-Planck operator can be represented by a finite matrix

the payback

claim:

optimal partition hypothesis

- the best of all possible state space partitions
- optimal for the given noise

the payback

claim:

optimal partition hypothesis

- optimal partition replaces stochastic PDEs by finite, low-dimensional Fokker-Planck matrices

the payback

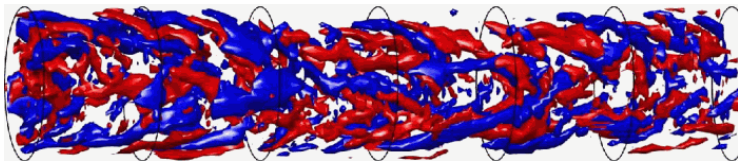
claim:

optimal partition hypothesis

- optimal partition replaces stochastic PDEs by finite, low-dimensional Fokker-Planck matrices
- finite matrix calculations, finite cycle expansions \Rightarrow optimal estimates of long-time observables (escape rates, Lyapunov exponents, etc.)

summary

- Computation of unstable periodic orbits in high-dimensional state spaces, such as Navier-Stokes,



is at the border of what is feasible numerically, and criteria to identify finite sets of the most important solutions are very much needed. Where are we to stop calculating orbits of a given hyperbolic flow?

summary

- Intuitively, as we look at longer and longer periodic orbits, their neighborhoods shrink exponentially with time, while the variance of the noise-induced orbit smearing remains bounded; there has to be a *turnover time*, a time at which the noise-induced width overwhelms the exponentially shrinking deterministic dynamics, so that no better resolution is possible.

summary

- We have described here the *optimal partition hypothesis*, a new method for partitioning the state space of a chaotic repeller in presence of weak Gaussian noise, and tested the method in a 1-dimensional setting against direct numerical Fokker-Planck operator calculation.

references

- D. Lippolis and P. Cvitanović, *How well can one resolve the state space of a chaotic map?*, Phys. Rev. Lett. 104, 014101 (2010); [arXiv.org:0902.4269](https://arxiv.org/abs/0902.4269)
- D. Lippolis and P. Cvitanović, *Optimal resolution of the state space of a chaotic flow in presence of noise (in preparation)*