Chapter 12

Conclusions and new challenges

The long and detailed analysis of symbolic dynamics in a variety of systems developed above, shows that symbolic dynamics may be a powerful tool for solving various problems in chaotic systems. Possibly this kind of detailed analysis of chaotic systems overshoots, and becomes too complex for a practical application to physical problems such as of finding the energy levels of a bound quantum system. Only future investigations can decide whether such a detailed description of the geometry of orbits is really necessary for practical calculations in different systems. If this turns out to be necessary, we believe that the symbolic description given here is a natural and effective way implementing this description.

Regardless of what the applications might turn out to be, we believe that the symbolic description of bifurcations and the pruning fronts discussed above is important for theoretical understanding of dynamical systems. In mathematical literature the discussion is usually centered around unimodal one-dimensional maps and the renormalization of bifurcations. For two-dimensional diffeomorphisms and flows the literature deals mostly with the problems of the existence of complete Smale horseshoe non-wandering sets, bifurcations creating homoclinic tangency points, the Newhouse theorem, and the different bifurcations of stable orbits [100, 198]. Not much has been done in systematically describing the resonance structure in the parameter space for multi-modal one-dimensional endomorphisms, and chaotic two-dimensional diffeomorphisms. For billiards the interest in mathematical literature has been on proving the ergodicity [32, 39, 181, 182, 200] and not on the understanding of bifurcations. For some billiards the existence of countable Markov partitions has been showed [34, 128, 129], but not how to obtain useful approximations to this partition. We think that the methods developed here to generalize the MSS and Milnor-Thurston theory lead to a better understanding of bifurcation structures in the parameter space, but this thesis does not pretend to be mathematically rigorous. The conjecture of the existence of a unique way to assign symbols to a pruned folded map is one important case where a proof, a counterexample or an improved conjecture is needed. The existence of pruning fronts in the billiard systems may be turned into theorems without too much effort, but the conjecture of existence of a pruning front for smooth Hamiltonian systems may be very difficult to prove. The main difficulty is constructuion of a "singular orbit" in a smooth potential that would be analog to the singular orbit in the billiard. We have pointed out the close connection between the bifurcation trees in the billiards and in the smooth potentials. We expect a "singular orbit" in a smooth potential to be at, or close to, a homoclinic/hetroclinic tangency with a geometric shape in the configuration space similar to a singular billiard orbit, e.g. an orbit tangent to a wall or bouncing off a singular point of the billiard wall. This is an important question which requires further investigation. Also it will be nontrivial to prove that the well-ordered symbols of the billiard are ordered the same way in the phase space of a smooth potential, a necessary prerequisite for construction of a monotone pruning front.

The transformation of a pruned region into an expansion of the zeta function is obtained by constructing a Markov transition graph and finding loops in this graph. This is faster than a direct construction of a matrix from the symbol plane, but finding all combinations of loops is time consuming for large graphs. The diagrams we construct are also not necessarily the smallest possible and their form may depend on the order in which the forbidden strings are implemented as we construct the graph. This implies that the "fundamental part" of our the zeta function expansions may be unnecessarily large. Our approximations to the pruning front are implemented by removing rational rectangles, and for billiard systems this seems not very efficient. An approximation with triangles would converge much better, but we do not know how to implement this to get the admissible orbits. In addition Markov graphs might have symmetries which should be removed. Symmetry decomposition for complete alphabets has been discussed in detail by Cvitanović and Eckhardt [49], but should also be implemented for the pruned systems.

Another question receiving interest lately is the question of the monotonicity of bifurcations in different maps. Recent work by Milnor, Tresser and others [59, 149, 151, 168] has shown that in bimodal maps with bifurcation diagrams similar to the symbolic parameter planes discussed in section 2.1, there are simple paths along which the topological entropy increases monotonically, and regions with constant entropy are connected in the parameter plane. Numerics indicate that these results also hold for the parameter spaces of polynomial maps [151]. The question of anti-monotonicity has been discussed by Yorke and coworkers [60, 125], who claim that bimodal and more complicated maps are anti-monotone. There has not been further work for higher modal maps following the aproach of by Milnor and Tresser, but the bifurcation diagrams obtained here, figures 2.13, 2.17 and 2.20, indicate that the results for bimodal maps also apply to three-modal and possibly higher-modal maps. However, in the higher modal maps bifurcations can change the modality and make the picture more complicated. We have also obtained similar symbolic parameter spaces for the once-folding maps, and we hope that this will enable us to prove monotonicity for once-folding maps as well. This however appears to disagree with the results of Kan, Koçak and Yorke [125]. The folding maps are complicated systems with an infinite-dimensional parameter spaces. Much more work is required before we can claim this problem to be solved, but we hope the work presented here is a step in the right direction.

Another problem of interest is the description of the parameter values for which the entropy changes from 0 to a positive number called the "border of chaos" [140]. By using symbolic parameter spaces this border may be described for the N-modal one-dimensional maps and for folding two-dimensional maps. This border may then also be understood in an ordinary parameter space. This is also a question for further studies.

The question of a chaotic attractor for a finite measure of parameter values has been addressed, and positive results have been obtained for the logistic map [21] and for the Hénon map with very small values for the parameter b [22]. Nothing is known about this question in bimodal and other more complicated maps.

The construction of symbols for an arbitrary smooth dynamical system is not understood. We believe that in a number of Hamiltonian systems we can use symbols defined for a corresponding billiard, but there currently exists no method for constructing symbols for any system. This is a difficult problem but progress here will be very interesting. See also ref. [68].

The method of Biham and Wenzel [26, 27] for finding periodic orbits in the Hénon map is interesting, and could be generalized to other once-folding maps and maybe to n-folding maps. With more than one folding the method will depend on the starting conditions, and an investigation of such methods may be interesting. We have shown here how the convergence of the BW method is closely related to the modality of the 1-dimensional approximation, and we expect this also for a generalized BW method.

In the field of quantum chaos there is a number of interesting questions under investigation. The quantum verions of our classical chaotic billiard systems are "particle in box" problems and the systems studied most. The semi-classical work on the three-body problem in atomic systems is very promising, and the results already obtained for these systems under special conditions are impressive [70, 199]. Further studies along this path may be the most exciting projects of the near future. The question of diffusion in classical and quantum systems is an interesting, but difficult problem [9], and application to astrophysical problems are interesting [20, 64].