1 Systematic Detection of Unstable Periodic Orbits in Discrete Chaotic Dynamical Systems

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1.1 Motivation

Unstable periodic orbits (UPOs) are the skeleton of complex chaotic systems In many areas of physics they allow the calculation of characteristic quantities of the underlying dynamics: Cycle expansion methods have been studied extensively for low-dimensional model systems as well as for experimental time series. The series expansion of semiclassical properties of classically chaotic Hamiltonian systems with respect to the length and stability coefficients of the periodic orbits is a fruitful and frequently applied technique to investigate properties of a quantum system. Last but not least UPOs are of particular importance for the control of chaotic dynamical systems.

Therefore it is not surprising that much effort has been given to develop techniques to find the periodic orbits of a given chaotic dynamical system.

Why are periodic orbits of chaotic dynamical systems so difficult to detect? One major difficulty is the exponential proliferation of the number of cycles with increasing period, which is an obstacle for the separation of UPOs once you have a set of them. The other prominent reason is the instability of the periodic orbits, i.e. the instability of the fixed points of the iterated map. Even more sophisticated Newton–Raphson (NR) methods rely on a good initial guess for the starting point. In general, you must be sure to be in the linear neighbourhood of the UPO. For longer orbits, this neighbourhood shrinks with the proliferation of the periodic orbits which clutter the bounded phase space volume of the attractor. Using NR methods, you lose track of the periodic orbit as soon as you leave this narrow domain of controlled linear dynamics, and the trajectory wanders around in a random–like walk. By chance you may hit the small domain of attraction of another periodic orbit, but more likely the trajectory will leave the attractor and diverge. Therefore these NR–algorithms rapidly become expensive and are limited to relatively short periods and low–dimensional systems.

A variety of other methods has been developed which focus either on time series analysis or on finding UPOs for given equations of motions [1, 2, 3, 4, 5, 6]. For a special class of systems, a numerical technique for calculating arbitrarily long UPOs to any desired accuracy was introduced in [6] for the Hénon map [7] and later applied to a few other dynamical systems [8, 9, 10]. This method allows the systematic computation of all UPOs of any given order, each given by a unique binary symbol sequence.

1.2 Basic Ideas

The above considerations show that periodic orbits would be more easy to detect if they were dissipatively stable. We developed a method [11] following this path for the calculation of UPOs of fully chaotic time–discrete systems. The idea is to transform the original dynamical system into a new one with the periodic orbits keeping their positions but changing their stability properties. For a particular transformed system a certain fraction of the periodic cycles becomes stable and can be found by simply iterating the transformed system. This set of stabilised points depends on the transformation used.

Let us first provide a description of the method based on the above introduced ideas. Consider a discrete chaotic dynamical system

$$U: \; \mathbf{r}_{i+1} = \mathbf{f}(\mathbf{r}_i) \tag{1}$$

in n dimensions. The fixed points (FPs) of the pth iterate $\mathbf{f}^{(p)}$ are points of the UPOs of period p. To find the FPs of U we transform U into different discrete dynamical systems, employing the following strategy: A set of transformations is specified which transforms U into new dynamical systems $\{S_{k\sigma}\}$ with the FPs keeping their original locations in space. The transformed systems have therefore stationary points at the positions of the FPs in the original system U. The double index $(k\sigma)$ refers to a labelling of the set of all transformations and will be explained below. The set of transformations is complete in the sense that for each unstable FP α_u of U there exists a specific transformed system $S_{k'\sigma'}$ of the set $\{S_{k\sigma}\}$ in which this particular FP is dissipatively stable (i.e. $|\lambda_i| < 1$ for eigenvalues λ_i). These stable stationary points can therefore be detected by simply iterating properly chosen starting points of the transformed system $S_{k'\sigma'}$. For each $S_{k'\sigma'}$ a different set of FPs of U is stabilised. The complete set of all stationary points of all transformed systems employing all possible transformations corresponds to the complete set of FPs of the original system U. We can find a minimal set of transformations $I_{\min}$ with the property that there exists for each unstable FP α_u at least one transformation in $I_{\min}$ in which the stationary point corresponding to α_u is stable. This minimal set $I_{\min}$ is the same for an arbitrary period p. The search for the FPs of U is now straightforward: A starting point chosen in the global neighbourhood (see below) of the FP α_u iterated with the transformed dynamical system $S_{k'\sigma'}$, $(k'\sigma') \in I_{\min}$ converges, due to the stability of α_s, to the position of

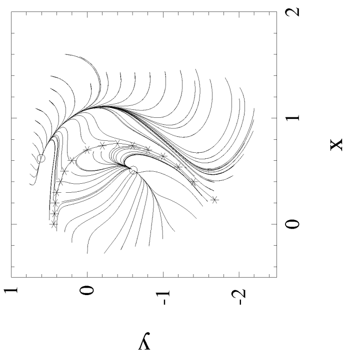

Figure 1: Ensemble of trajectories of the transformed map according Eqn. (2) for the stabilisation of the period–2 orbit of the Ikeda attractor The circles indicate the position of the points of the period–2 orbit, and the line connecting the crosses indicates the border line between the two basins of attraction.

α_s which is equal to that of α_u. As an example, fig. 1 shows a set of trajectories which converge in the stabilised period–2 orbit of the Ikeda map. Numerical investigations show that the domain of attraction of a stabilised FP is a rather extended and connected area and by no means confined to the linear neighbourhood (see fig. 1). For certain cases, the basin of attraction contains the complete phase space of the attractor, so that one can be sure to be within the attracting basin of a FP if you place the initial condition anywhere on the attractor. Let us now specify the systems $S_{k\sigma}$:

$$S_{k\sigma}: \quad \mathbf{r}_{i+1} = \mathbf{r}_i + \lambda \cdot \mathbf{C}_{k\sigma}[\mathbf{f}^{(p)}(\mathbf{r}_i) - \mathbf{r}_i] \tag{2}$$

$S_{k\sigma}$ are linear transformations of the original dynamical law U. $\mathbf{C}_{k\sigma}$ are invertible constant $n \times n$ matrices. The definition of $S_{k\sigma}$ and U clearly shows that their FPs are one to one and at the same positions. Eqn. (2) implies the following relation for the stability matrices $\mathbf{T}_U$ and $\mathbf{T}_{S_{k\sigma}}$ of U and $S_{k\sigma}$, respectively:

$$\mathbf{T}_{S_{k\sigma}} = \mathbf{1} + \lambda \mathbf{C}_{k\sigma}(\mathbf{T}_U - \mathbf{1}) \tag{3}$$

In ref. [11] we show that it is possible to choose the set of matrices $\{\mathbf{C}_{k\sigma}\}$ such that it contains orthogonal matrices with only one non-vanishing entry ± 1 per row or column. These matrices operating on the difference vector $\mathbf{f}^{(p)}(\mathbf{r}) - \mathbf{r}$ represent a group of special reflections and permutations. The double index $(k\sigma)$ is related to certain properties of the corresponding matrix, which are of importance for their role with respect to the stabilisation of FPs. $\sigma = \pm$ indicates the sign of the matrix determinant, and k is an additional label to uniquely specify the matrices. For more details see ref. [12]. This reference also contains some pictorial sketches of the change in geometry a FP undergoes if it is transformed using a particular $\mathbf{C}_{k\sigma}$. In two dimensions we have

$$\mathbf{C}_{0+} = -\mathbf{C}_{2+} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{C}_{0-} = -\mathbf{C}_{2-} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\mathbf{C}_{1+} = -\mathbf{C}_{3+} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathbf{C}_{1-} = -\mathbf{C}_{3-} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

The minimal set $I_{\min}$ that suffices to stabilise all kinds of possible FPs in a two dimensional fully chaotic system is significantly smaller than the set of pairs $(k\sigma)$ belonging to the matrices $\{S_{k\sigma}\}$. In two dimensions the minimal sets is either $\{\mathbf{C}_{0+}, \mathbf{C}_{0-}, \mathbf{C}_{2-}\}$ or $\{\mathbf{C}_{0+}, \mathbf{C}_{1-}, \mathbf{C}_{3-}\}$. Given a certain unstable FP α_u the above choice of linear transformations induced by the set $\{\mathbf{C}_{k\sigma}\}$ of matrices allows to find a particular $\mathbf{C}_{k'\sigma'}$ such that the real parts of the eigenvalues of $\mathbf{C}_{k'\sigma'}(\mathbf{T}_U - \mathbf{1})$ are negative at the position of the FP.

We gain new insight into the geometrical interpretation of the method of stability transformations if we consider the limit $\lambda \to 0$. In this limit, the transformed system is a time–continuous one, given by

$$S_{k\sigma}: \quad \dot{\mathbf{r}} = \mathbf{C}_{k\sigma}[\mathbf{f}^{(p)}(\mathbf{r}) - \mathbf{r}] \tag{4}$$

The relation of the stability matrices is

$$\mathbf{T}_{S_{k\sigma}} = \mathbf{C}_{k\sigma}(\mathbf{T}_U - \mathbf{1}) \tag{5}$$

You can regard the different stability transformations as linear functions of the different vector fields given by the difference vector $\mathbf{f}^{(p)}(\mathbf{x}) - \mathbf{x}$. The set of these linear functions is nothing but the group of permutations and sign inversions of the different coordinates of this vector. You can distinguish different types of FPs with respect to the direction of this difference vector relative to the FP. Each of these types require a different matrix $\mathbf{C}_{k\sigma}$ to stabilise the FP. This is achieved by rotating and reflecting the corresponding difference vector, so that the resulting vector, now considered as a flux, is directed to the FP itself.

1.3 Numerical Implementation

For an effective numerical implementation we use the Euler-discretised form (2) of the transformed system. λ is the step size of the discretisation. If λ is chosen sufficiently small, the magnitude of the real parts of the eigenvalues of the FP α_s in the transformed system are smaller than one and we therefore encounter a stable FP which can be detected as described above. The critical value of λ, which just suffices to make the FP stable, can immediately be read off from the quadratic equations relating the stability coefficients of the original system U and those of the transformed system $S_{k\sigma}$ [11]. The above procedure can easily be extended to higher iterates $\mathbf{f}^{(p)}(\mathbf{r})$ of U (by replacing $\mathbf{f}$ with $\mathbf{f}^{(p)}$ in Eq. (2)) allowing us to determine all order p cycles of U. The step size then works as a upper cut–off for the stability of the FPs which are stable in the corresponding transformed and discretized dynamical system.

Due to its global character, this method is well suited for numerical investigations of a given dynamical system. Even points far from the linear neighbourhood of a FP are attracted close to the FP after a finite number of iterations of the transformed dynamical law. The basin of attraction of a single stabilised FP is a simply connected area in phase space. For the particular case of stabilisation using the matrix $\mathbf{C}_{0+}$ any point in the phase space turns out to be in the domain of attraction of a FP of the attractor. For the other two matrices of each minimal set the domains of attraction typically fill approx. 60% to 80% of the attractor. Since the size of the areas of attraction of the different FPs are distributed rather uniformly in phase space, the typical number of starting points needed to obtain the UPOs of a given length on the attractor is only slightly more than the expected number of cycle points themselves.

If one is interested in the complete set of FPs of a given period, the best way to proceed is as follows. We start with a series of sets of starting points which are cumulative in the following sense: Each set is constructed such that it fills the gap larger than a given size in the union of all preceding sets. The size of the individual sets can be of the order of 100 to 1000, depending on the number of FPs one expects to find. The starting points of each set are propagated with the transformed maps S_{0+}, S_{0-}, and S_{2-}, i.e. applying the matrices $\mathbf{C}_{0+}$, $\mathbf{C}_{0-}$ and $\mathbf{C}_{2-}$ according to (2). One can find the optimal λ to be used for the propagation by propagating the dynamical system stabilised with S_{0+}. In general, all starting points converge on the attractor provided λ is small enough. If this is not the case, the trajectory either diverges (if λ is far to large) or it oscillates back and forth in a small section of the phase space (if λ is close to its stabilising value). On the other hand, it should not be chosen too small, since this slows down the speed of convergence remarkably.

The step size $|\mathbf{r}_{i+1} - \mathbf{r}_i|$ decreases exponentially when the trajectory approaches the FP. To get the coordinates of the FPs with a high precision, one therefore needs a large number of iterations for the trajectory which is already in the linear neighbourhood of the FP. To speed up the final part of the convergence we therefore suggest to combine the above described approach of stabilising transformations with an algorithm which converges fast in the linear neighbourhood. A simple Newton algorithm does a good job. Each, say, 1000 iterations of the above approach according to (2) the Newton algorithm is started. One then has to check whether it converges to the desired FP or not.

Using the above procedure the different cumulative sets of initial points are propagated. The result of propagating the sets is a different set of FPs each time. You may now check whether in a specific set you found FPs which have not occurred in any of the sets before. In this way you can check the convergence of your set of FPs. This is of course not a rigorous way of checking the completeness but an empirical one.

Following the above you get many FPs which belong to the same periodic orbit. To keep only one FP per periodic orbit we found it helpful to calculate the distance d_{xy} between any two orbits of period p, $\mathbf{x_i} = (x_{1i}, x_{2i})^T$, $\mathbf{y_i} = (y_{1i}, y_{2i})^T$, $i = 1, \dots, p$: $d_{xy} = \min_{\substack{k=0,\dots \\ \dots,p-1}} \sum_{\substack{1\le i\le p \\ 1\le j\le 2}} (x_{ji} - y_{j(i+k \bmod p)})^2$ By forming all possible pairs of any two FPs of this set and looking at the resulting distribution of values of d_{xy} (e.g. by plotting all d_{xy} in a logarithmic scale) one can distinguish three subsets, separated by gaps of several orders of magnitude. The set with the largest values of d_{xy} contains all pairs formed with different orbits, whereas the second and third largest set are composed of pairs of identical orbits. This way you can distinguish identical from close–by, but separate orbits. If the first and the second zone happen to merge, the accuracy of the FPs is not sufficient to distinguish different orbits and has to be refined by further propagation. Recently a modification of the above method has been suggested [17]. In this approach, each step of the iteration scheme is a linear superposition of a step of the stability transformed system (2) and a step of a Newton–Raphson algorithm. Far from the linear neighbourhood the weight is dominantly on the globally acting stability transformation algorithm (2). Close to the FP the steps of the iteration are dominated by the linear Newton–Raphson procedure. The relative weight is changed in a way that it allows the calculation of a large number of FPs up to high periods, similar to the previously discussed method.

1.4 Stability Ordering

The parameter λ, the step size of the discretisation eqn. (2), is a key quantity here. It is related to the stability of the desired cycle in the transformed system: The more unstable a FP is, the smaller λ has to be to stabilise it.

With increasing period of the cycles, the unstable eigenvalue of the stability matrix increases and therefore λ has to be reduced to achieve stabilisation of all FPs. In many cases the least unstable periodic orbits of a given period p are of particular interest [13]. In this context λ operates as a stability filter. It allows the selective stabilisation of only those UPOs which possess Lyapunov exponents smaller than a critical value. If one starts the search for UPOs within a certain period p with a value $\lambda \cong O(10^{-1})$ and gradually lowers λ one obtains the sequence of all unstable orbits of order p sorted with increasing values of their Lyapunov exponents. For the specific choice $(k\sigma) = (0+) \in I_{\min}$ the relation between λ and the stability coefficients of the FPs of the original system U is strictly monotonous. Transformed dynamical systems $S_{k\sigma}$ belonging to other pairs $(k'\sigma') \neq (0+)$ do not obey such a strict behaviour but show a rough ordering of the sequence of stability eigenvalues of the FPs of U stabilised in the course of decreasing values for λ. Stability ordered periodic orbits might be of interest for cycle expansions of dynamical quantities of chaotic systems without a well known symbolic dynamics. For these systems, an ordering of periodic orbits with respect to their stability has been suggested [14, 15, 16]. It gives good results if you consider only the periodic orbits up to a given cut–off value of their stability.

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