Exercise 3.1: A pinball simulator

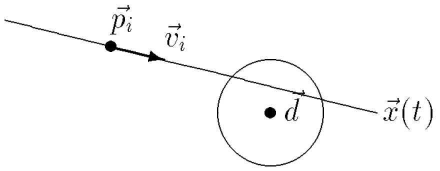
Implement the disk $\rightarrow$ disk maps to compute a trajectory of a pinball for a given starting point, and given $R : a$ ratio. Fast code will use elementary geometry (only one $\sqrt{\cdots}$ per iteration, rest are multiplications) and eschew trigonomoetric functions. Provide a graphic display of the trajectories and of the Poincaré section iterates.

Outline of Algorithm

The pinball simulator calculates the trajectory of a ray a light bouncing around three disks. To calculate the trajectory we need to know if and when the ray of light will hit a disk. If the ray of light is aligned in the direction of two disks then clearly it will hit the disk which is closest. We suppose that after i bounces the light ray is at a position $\vec{p}_i$ and has a velocity $\vec{v}_i$ (which by construction is always normalized, $|\vec{v}_i| = 1$. The trajectory of the light will then lie on the line

$$\vec{x}(t) = \vec{p}_i + \vec{v}_i t \tag{1}$$

Schematically this is shown below



To find if and when the ray collide with a disk at position $\vec{d}$ we calculate when the ray is at a distance a from the disk centre. This occurs when

$$|\vec{x}(t) - \vec{d}|^2 = |(\vec{p}_i - \vec{d}) - \vec{v}_i t|^2 = a^2 \tag{2}$$

defining $\vec{y} = \vec{p}_i - \vec{d}$ then the line intersects the perimeter of the disk at times

$$t_\pm = \vec{v}_i \cdot \vec{y} \pm \sqrt{|\vec{v}_i \cdot \vec{y}|^2 - |\vec{y}|^2 + a^2} \tag{3}$$

Thus the condition for a potential collision is that $\vec{v}_i \cdot \vec{y}|^2 - |\vec{y}|^2 + a^2 > 0$. We are only interested in times $t > 0$ which provides a supplementary condition $\vec{d} \cdot \vec{y} > 0$ (since the disk is convex). The solution t_- clearly corresponds to the true collision time provided it is small than that for any other disk. The collision point occurs at

$$\vec{p}_{i+1} = \vec{p}_i + \vec{v}_i t_- \tag{4}$$

Defining the vector $\vec{n}$ to be perpendicular to the disk at the collision point $\vec{p}_{i+1}$, so that

$$\vec{n} = \frac{\vec{p}_{i+1} - \vec{d}}{|\vec{p}_{i+1} - \vec{d}|} = \frac{1}{a}(\vec{p}_{i+1} - \vec{d}) \tag{5}$$

then the new velocity is given by

$$\vec{v}_{i+1} = \vec{v}_i - 2(\vec{v}_i \cdot \vec{n})\vec{n} \tag{6}$$