

ChaosBook.org chapter
flows

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dynamical systems

state space (often: phase space)

$\mathcal{M} \in \mathbb{R}^d$: d numbers determine the state of the system

globally, \mathcal{M} is a manifold - a torus, a cylinder, . . .

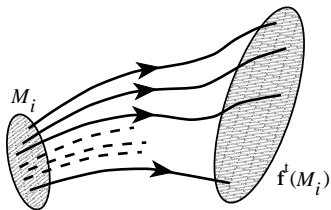
representative point

$x(t) \in \mathcal{M}$: a state of physical system at instant in time

dynamics

map $f^t(x_0)$ = representative point time t later

evolution rule



f^t maps a region \mathcal{M}_i of the state space into the region $f^t(\mathcal{M}_i)$.

smooth dynamical system

f^t can be differentiated as many times as needed

deterministic dynamics

evolution rule f maps a point into exactly one point at time t

dynamical system

the pair (\mathcal{M}, f)

dynamical systems

flow

evolution in continuous time $t \in \mathbb{R}$:

iteration of a map

$$x_{n+1} = f(x_n)$$

evolution advances in discrete time steps, integer time $n \in \mathbb{Z}$

flows

for infinitesimal times, flows can be defined by differential equations - a generalized **vector field**

$$v(x) = \dot{x}(t)$$

examples

Newton's laws for a mechanical system

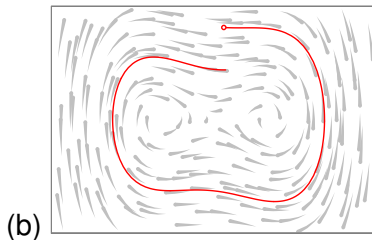
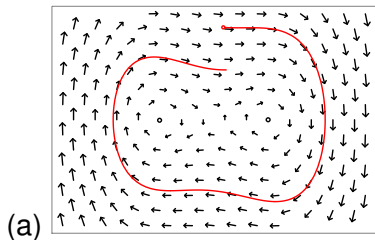
general flows, mechanical or not, defined by a time-independent vector field $v(x)$

example: unforced Duffing system

a two-dimensional vector field $v(x)$

$$\dot{x}(t) = y(t)$$

$$\dot{y}(t) = -0.15y(t) + x(t) - x(t)^3$$



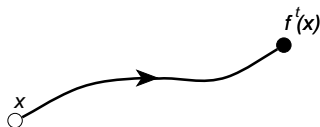
two visualization of the velocity vector field $v(x)$, superimposed over the configuration coordinates $(x(t), y(t)) \in \mathcal{M}$

velocity field belongs to a different space,

the **tangent bundle** $\mathbf{T}\mathcal{M}$

orbit vs. trajectory

trajectory



curve $x(t) = f^t(x)$ through the point x traced out by the evolution rule f^t

after a time t the point is at $f^t(x)$

orbit of x_0

subset in \mathcal{M} of points reached by the (possibly infinite) trajectory of x_0

For a flow, an orbit is a continuous curve; for a map, it is a sequence of points

orbit is a dynamically invariant set

orbit, or a solution of x_0 :

subset of points $\mathcal{M}_{x_0} \subset \mathcal{M}$ that belong to the infinite-time trajectory of a given point x_0

\mathcal{M}_{x_0} is a **dynamically invariant** set, the totality of states that can be reached from x_0 , with the full state space \mathcal{M} foliated into a union of such orbits

we label a generic orbit \mathcal{M}_{x_0} by a point belonging to it, for example $x_0 = x(0)$

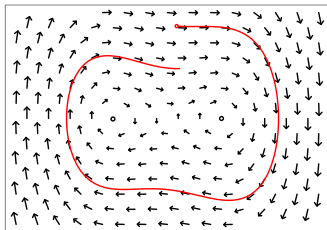
possible trajectories?

- stationary: $f^t(x) = x$ for all t
- periodic: $f^t(x) = f^{t+T_p}(x)$ for a given minimum period T_p
- aperiodic: $f^t(x) \neq f^{t'}(x)$ for all $t \neq t'$.

example : Duffing flow equilibria

x_q is an **equilibrium point**

if $v(x_q) = 0$



Duffing flow is bit of a bore: every trajectory ends up in one of the two attractive equilibrium points

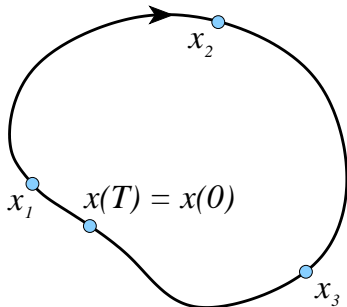
periodic orbits

periodic point

returns to the initial point after a finite time, $x = f^{T_p}(x)$

periodic orbit

p is the set of periodic points
 $p = \mathcal{M}_p = \{x_1, x_2, \dots\}$ swept out by the trajectory of any one of them in the finite time T_p



periodic orbits - a very small subset of the state space, in the same sense that rational numbers are a **set of zero measure** on the unit interval

generic orbit might be ergodic, unstable and essentially uncontrollable

ChaosBook strategy : “geometry of chaos”

populate the state space by a hierarchy of compact invariant sets (equilibria, periodic orbits, invariant tori, ...), each computable in a finite time

together with their invariant stable/unstable manifolds

orbits which are compact invariant sets we label by alphabet convenient in a particular application:

- $EQ = x_{EQ} = \mathcal{M}_{EQ}$ for an equilibrium
- $p = \mathcal{M}_p$ for a periodic orbit
- \mathcal{M}_T for an invariant torus
- ...

close recurrences

for a generic dynamical system most motions are **aperiodic**, so give up on exact periodicity, consider instead **close recurrences**

non-wandering set

point x is **recurrent** or **non-wandering** if for any open neighborhood \mathcal{M}_0 of x there exists a later time t such that

$$f^t(x) \in \mathcal{M}_0$$

Ω : the **non-wandering set** of f , the union of all the non-wandering points of \mathcal{M}

Ω is the key to understanding the long-time behavior of a dynamical system

attractor

if there exists a connected state space volume that maps into itself under forward evolution, the flow is globally contracting onto a subset of \mathcal{M} , the **attractor**

there can coexist any number of distinct attracting sets, each with its own **basin of attraction**, the set of all points that fall into the attractor under forward evolution

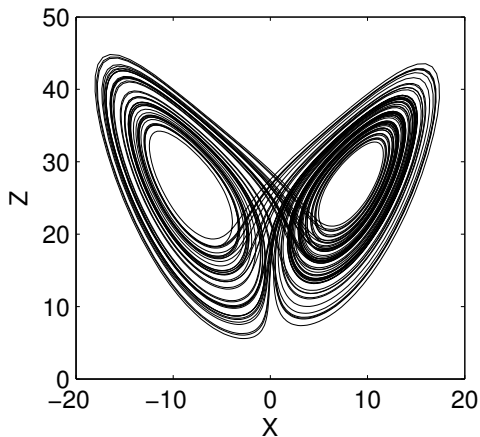
the attractor can be

- a fixed point
- a periodic orbit
- aperiodic
- or any combination of the above

strange attractor

the most interesting case is that of an aperiodic recurrent attractor, to which we shall refer loosely as a **strange attractor**

example : Lorenz strange attractor

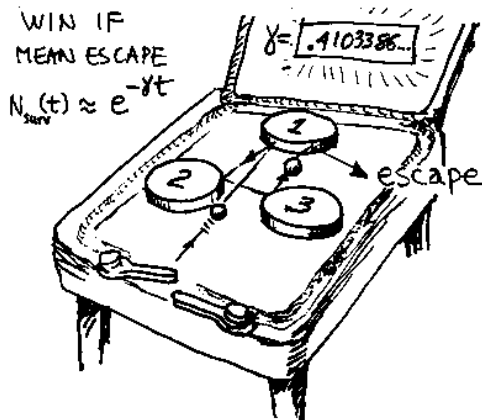


Edward Lorenz: the weather will remain unpredictable

strange repeller

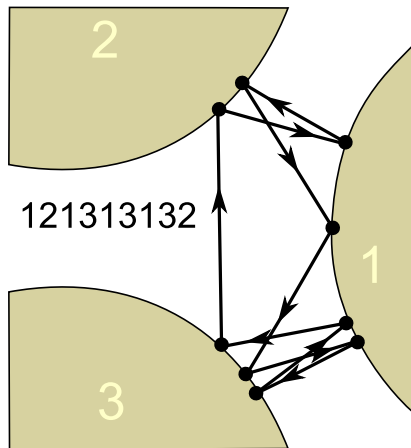
conversely, if we can enclose the non-wandering set Ω by a connected state space volume \mathcal{M}_0 and then show that almost all points within \mathcal{M}_0 , but not in Ω , eventually exit \mathcal{M}_0 , we refer to the non-wandering set Ω as a **repeller**

example : game of pinball



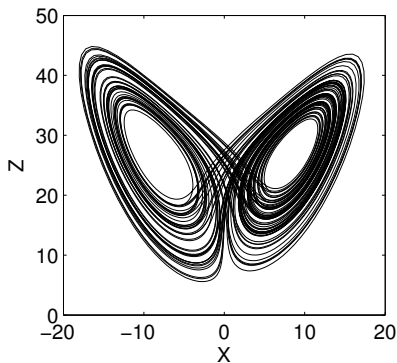
Pinball Wizard: no pinball bounces forever

periodic orbit: a perfect pinball shot



Pinball Wizard: cannot make it, it is an unstable solution

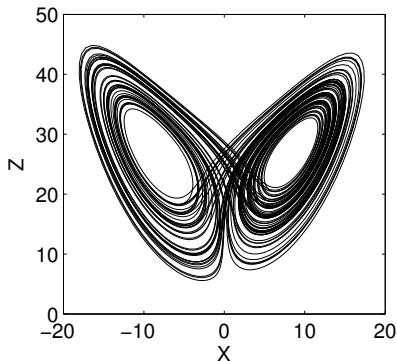
example : Lorenz “butterfly” strange attractor



$$\dot{x} = v(x) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma(y - x) \\ \rho x - y - xz \\ xy - bz \end{bmatrix}$$

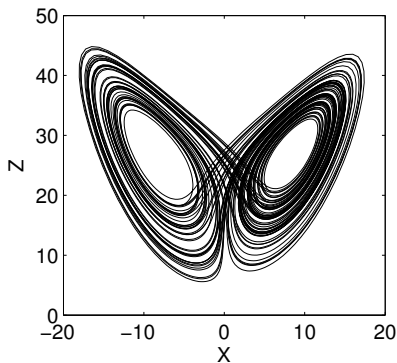
Lorenz fixed $\sigma = 10$, $b = 8/3$, varied the “Rayleigh number” ρ

example : Lorenz “butterfly” strange attractor



$0 < \rho < 1$ the equilibrium $EQ_0 = (0, 0, 0)$ attractive

example : Lorenz “butterfly” strange attractor



at $\rho = 1$ a pitchfork bifurcation into a pair of equilibria at

$$x_{EQ1,2} = (\pm\sqrt{b(\rho - 1)}, \pm\sqrt{b(\rho - 1)}, \rho - 1),$$

example : Rössler flow

Rössler flow

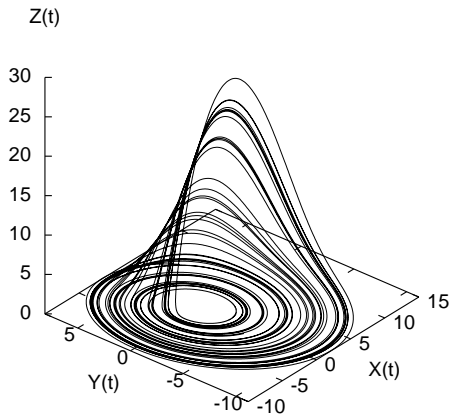
$$\dot{x} = -y - z$$

$$\dot{y} = x + ay$$

$$\dot{z} = b + z(x - c),$$

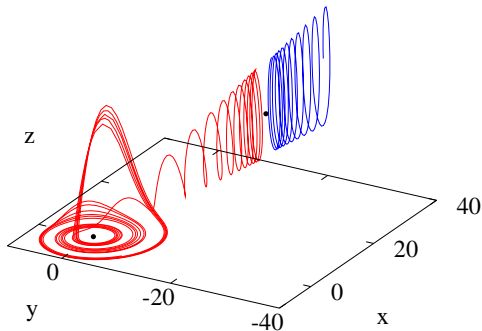
$$a = b = 0.2, \quad c = 5.7$$

typical numerically integrated
long-time trajectory



equilibria of Rössler flow

two trajectories of the Rössler flow initiated in the neighborhood of the “upper” equilibrium point

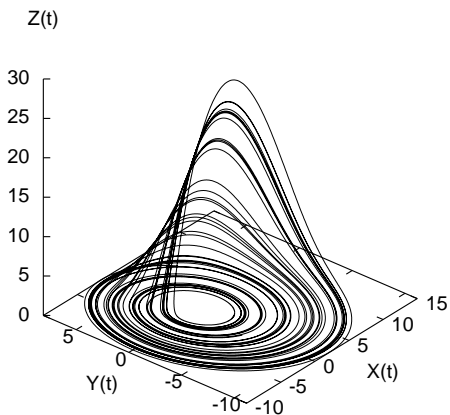


2 repelling equilibrium points (no dynamics there!)

$$(x^-, y^-, z^-) = (0.0070, -0.0351, 0.0351)$$

$$(x^+, y^+, z^+) = (5.6929, -28.464, 28.464)$$

a strange attractor?



a trajectory of the
Rössler flow up to
time $t = 250$

trajectories that start out sufficiently close to the origin seem to converge to a **strange attractor**

Résumé

A **dynamical system** – a flow, or an iterated map – is defined by specifying

a pair

$$(\mathcal{M}, f)$$

where \mathcal{M} is the state space, and $f : \mathcal{M} \rightarrow \mathcal{M}$

the key concepts in exploration of the long time dynamics are the notions of

- **recurrence** and of
- **non-wandering set** of f , the union of all the non-wandering points of \mathcal{M}