Appendix G

Transport of vector fields

Man who says it cannot be done should not interrupt man doing it.
—Sayings of Vattay Gábor

In this appendix we show that the multidimensional Lyapunov exponents and relaxation exponents (dynamo rates) of vector fields can be expressed in terms of leading eigenvalues of appropriate evolution operators.

G.1 Evolution operator for Lyapunov exponents

Lyapunov exponents were introduced and computed for 1-d maps in sect. 15.3.2. For higher-dimensional flows only the fundamental matrices are multiplicative, not individual eigenvalues, and the construction of the evolution operator for evaluation of the Lyapunov spectra requires the extension of evolution equations to the flow in the tangent space. We now develop the requisite theory.

Here we construct a multiplicative evolution operator (G.4) whose spectral determinant (G.8) yields the leading Lyapunov exponent of a d-dimensional flow (and is entire for Axiom A flows).

The key idea is to extending the dynamical system by the tangent space of the flow, suggested by the standard numerical methods for evaluation of Lyapunov exponents: start at x_0 with an initial infinitesimal tangent space vector $\eta(0) \in \mathbf{T}\mathcal{M}_x$, and let the flow transport it along the trajectory $x(t) = f^t(x_0)$.

The dynamics in the $(x, \eta) \in U \times TU_x$ space is governed by the system of equations of variations [1]:

$$\dot{x} = \mathbf{v}(x) , \quad \dot{\eta} = \mathbf{D}\mathbf{v}(x)\eta .$$

Here $\mathbf{D}\mathbf{v}(x)$ is the derivative matrix of the flow. We write the solution as

$$x(t) = f^t(x_0) , \quad \eta(t) = M^t(x_0) \cdot \eta_0 , \tag{G.1}$$

with the tangent space vector η transported by the stability matrix $M^t(x_0) = \partial x(t)/\partial x_0$.

As explained in sect. 4.1, the growth rate of this vector is multiplicative along the trajectory and can be represented as $\eta(t) = |\eta(t)|/|\eta(0)|\mathbf{u}(t)$ where $\mathbf{u}(t)$ is a "unit" vector in some norm $||.||$. For asymptotic times and for almost every initial $(x_0, \eta(0))$, this factor converges to the leading eigenvalue of the linearized stability matrix of the flow.

We implement this multiplicative evaluation of stability eigenvalues by adjoining the d-dimensional transverse tangent space $\eta \in \mathbf{T}\mathcal{M}_x$; $\eta(x)\mathbf{v}(x) = 0$ to the $(d+1)$-dimensional dynamical evolution space $x \in \mathcal{M} \subset \mathbb{R}^{d+1}$. In order to determine the length of the vector η we introduce a homogeneous differentiable scalar function $g(\eta) = ||\eta||$. It has the property $g(\Lambda\eta) = |\Lambda|g(\eta)$ for any Λ. An example is the projection of a vector to its dth component

$$g\begin{pmatrix} \eta_1 \\ \eta_2 \\ \cdots \\ \eta_d \end{pmatrix} = |\eta_d| .$$

Any vector $\eta \in TU_x$ can now be represented by the product $\eta = \Lambda\mathbf{u}$, where $\mathbf{u}$ is a "unit" vector in the sense that its norm is $||\mathbf{u}|| = 1$, and the factor

$$\Lambda^t(x_0, \mathbf{u}_0) = g(\eta(t)) = g(M^t(x_0) \cdot \mathbf{u}_0) \tag{G.2}$$

is the multiplicative "stretching" factor.

Unlike the leading eigenvalue of the Jacobian the stretching factor is multiplicative along the trajectory:

$$\Lambda^{t'+t}(x_0, \mathbf{u}_0) = \Lambda^{t'}(x(t), \mathbf{u}(t))\, \Lambda^t(x_0, \mathbf{u}_0).$$

[exercise G.1]

The $\mathbf{u}$ evolution constrained to $E\mathbf{T}_{g,x}$, the space of unit transverse tangent vectors, is given by rescaling of (G.1):

$$\mathbf{u}' = R^t(x, \mathbf{u}) = \frac{1}{\Lambda^t(x, \mathbf{u})} M^t(x) \cdot \mathbf{u} . \tag{G.3}$$

Eqs. (G.1), (G.2) and (G.3) enable us to define a *multiplicative* evolution operator on the extended space $U \times E\mathbf{T}_{g,x}$

$$\mathcal{L}^t(x', \mathbf{u}'; x, \mathbf{u}) = \delta\big(x' - f^t(x)\big) \frac{\delta(\mathbf{u}' - R^t(x, \mathbf{u}))}{|\Lambda^t(x, \mathbf{u})|^{\beta-1}} , \tag{G.4}$$

where β is a variable.

To evaluate the expectation value of $\log|\Lambda^t(x,\mathbf{u})|$ which is the Lyapunov exponent we again have to take the proper derivative of the leading eigenvalue of (G.4). In order to derive the trace formula for the operator (G.4) we need to evaluate $\mathrm{Tr}\,\mathcal{L}^t = \int dx d\mathbf{u}\, \mathcal{L}^t(\mathbf{u},x;\mathbf{u},x)$. The $\int dx$ integral yields a weighted sum over prime periodic orbits p and their repetitions r:

$$\begin{aligned} \mathrm{Tr}\,\mathcal{L}^t &= \sum_p T_p \sum_{r=1}^{\infty} \frac{\delta\left(t - rT_p\right)}{|\det(1-M_p^r)|} \Delta_{p,r}, \\ \Delta_{p,r} &= \int_g d\mathbf{u}\, \frac{\delta\left(\mathbf{u} - R^{T_p r}(x_p,\mathbf{u})\right)}{|\Lambda^{T_p r}(x_p,\mathbf{u})|^{\beta-1}}, \end{aligned} \tag{G.5}$$

where M_p is the prime cycle p transverse stability matrix. As we shall see below, $\Delta_{p,r}$ is intrinsic to cycle p, and independent of any particular cycle point x_p.

We note next that if the trajectory $f^t(x)$ is periodic with period T, the tangent space contains d periodic solutions

$$\mathbf{e}_i(x(T+t)) = \mathbf{e}_i(x(t)), \quad i = 1, ..., d,$$

corresponding to the d unit eigenvectors $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_d\}$ of the transverse stability matrix, with "stretching" factors (G.2) given by its eigenvalues

$$M_p(x)\cdot\mathbf{e}_i(x) = \Lambda_{p,i}\,\mathbf{e}_i(x), \quad i = 1, ..., d. \qquad \text{(no summation on } i\text{)}$$

The $\int d\mathbf{u}$ integral in (G.5) picks up contributions from these periodic solutions. In order to compute the stability of the ith eigendirection solution, it is convenient to expand the variation around the eigenvector $\mathbf{e}_i$ in the stability matrix eigenbasis $\delta\mathbf{u} = \sum \delta u_\ell\, \mathbf{e}_\ell$. The variation of the map (G.3) at a complete period $t = T$ is then given by

$$\begin{aligned} \delta R^T(\mathbf{e}_i) &= \frac{M\cdot\delta\mathbf{u}}{g(M\cdot\mathbf{e}_i)} - \frac{M\cdot\mathbf{e}_i}{g(M\cdot\mathbf{e}_i)^2}\left(\frac{\partial g(\mathbf{e}_i)}{\partial\mathbf{u}}\cdot M\cdot\delta\mathbf{u}\right) \\ &= \sum_{k\neq i} \frac{\Lambda_{p,k}}{\Lambda_{p,i}}\left(\mathbf{e}_k - \mathbf{e}_i\frac{\partial g(\mathbf{e}_i)}{\partial u_k}\right)\delta u_k\,. \end{aligned} \tag{G.6}$$

The δu_i component does not contribute to this sum since $g(\mathbf{e}_i + du_i\mathbf{e}_i) = 1 + du_i$ implies $\partial g(\mathbf{e}_i)/\partial u_i = 1$. Indeed, infinitesimal variations $\delta\mathbf{u}$ must satisfy

$$g(\mathbf{u}+\delta\mathbf{u}) = g(\mathbf{u}) = 1 \quad \Longrightarrow \quad \sum_{\ell=1}^{d} \delta u_\ell \frac{\partial g(\mathbf{u})}{\partial u_\ell} = 0\,,$$

so the allowed variations are of form

$$\delta\mathbf{u} = \sum_{k\neq i}\left(\mathbf{e}_k - \mathbf{e}_i\frac{\partial g(\mathbf{e}_i)}{\partial u_k}\right)c_k\,, \quad |c_k| \ll 1\,,$$

and in the neighborhood of the $\mathbf{e}_i$ eigenvector the $\int d\mathbf{u}$ integral can be expressed as

$$\int_g d\mathbf{u} = \int \prod_{k\neq i} dc_k\,.$$

Inserting these variations into the $\int d\mathbf{u}$ integral we obtain

$$\begin{aligned} &\int_g d\mathbf{u} \quad \delta\left(\mathbf{e}_i + \delta\mathbf{u} - R^T(\mathbf{e}_i) - \delta R^T(\mathbf{e}_i) + \ldots\right) \\ &= \int \prod_{k\neq i} dc_k\, \delta((1-\Lambda_k/\Lambda_i)c_k + \ldots) \\ &= \prod_{k\neq i} \frac{1}{|1-\Lambda_k/\Lambda_i|}\,, \end{aligned}$$

and the $\int d\mathbf{u}$ trace (G.5) becomes

$$\Delta_{p,r} = \sum_{i=1}^{d} \frac{1}{|\,\Lambda_{p,i}^r\,|^{\beta-1}} \prod_{k\neq i} \frac{1}{|\,1-\Lambda_{p,k}^r/\Lambda_{p,i}^r\,|}\,. \tag{G.7}$$

The corresponding spectral determinant is obtained by observing that the Laplace transform of the trace (16.23) is a logarithmic derivative $\mathrm{Tr}\,\mathcal{L}(s) = -\frac{d}{ds}\log F(s)$ of the spectral determinant:

$$F(\beta,s) = \exp\left(-\sum_{p,r}\frac{e^{sT_p r}}{r\,|\det(1-M_p^r)|}\Delta_{p,r}(\beta)\right). \tag{G.8}$$

This determinant is the central result of this section. Its zeros correspond to the eigenvalues of the evolution operator (G.4), and can be evaluated by the cycle expansion methods.

The leading zero of (G.8) is called "pressure" (or free energy)

$$P(\beta) = s_0(\beta). \tag{G.9}$$

The average Lyapunov exponent is then given by the first derivative of the pressure at $\beta = 1$:

$$\overline{\lambda} = P'(1). \tag{G.10}$$

The simplest application of (G.8) is to 2-dimensional hyperbolic Hamiltonian maps. The stability eigenvalues are related by $\Lambda_1 = 1/\Lambda_2 = \Lambda$, and the spectral determinant is given by

$$
\begin{aligned}
F(\beta, z) &= \exp\left(-\sum_{p,r} \frac{z^{rn_p}}{r \mid \Lambda_p^r \mid} \frac{1}{(1-1/\Lambda_p^r)^2} \Delta_{p,r}(\beta) \right) \\
\Delta_{p,r}(\beta) &= \frac{\mid \Lambda_p^r \mid^{1-\beta}}{1-1/\Lambda_p^{2r}} + \frac{\mid \Lambda_p^r \mid^{\beta-3}}{1-1/\Lambda_p^{2r}} \,. \qquad \text{(G.11)}
\end{aligned}
$$

The dynamics (G.3) can be restricted to a u unit eigenvector neighborhood corresponding to the largest eigenvalue of the Jacobi matrix. On this neighborhood the largest eigenvalue of the Jacobi matrix is the only fixed point, and the spectral determinant obtained by keeping only the largest term the $\Delta_{p,r}$ sum in (G.7) is also entire.

In case of maps it is practical to introduce the logarithm of the leading zero and to call it "pressure"

$$P(\beta) = \log z_0(\beta). \qquad \text{(G.12)}$$

The average of the Lyapunov exponent of the map is then given by the first derivative of the pressure at $\beta = 1$:

$$\overline{\lambda} = P'(1). \qquad \text{(G.13)}$$

By factorizing the determinant (G.11) into products of zeta functions we can conclude that the leading zero of the (G.4) can also be recovered from the leading zeta function

$$1/\zeta_0(\beta, z) = \exp\left(-\sum_{p,r} \frac{z^{rn_p}}{r|\Lambda_p^r|^\beta} \right). \qquad \text{(G.14)}$$

This zeta function plays a key role in thermodynamic applications as we will will see in Chapter 22.

G.2 Advection of vector fields by chaotic flows

Fluid motions can move embedded vector fields around. An example is the magnetic field of the Sun which is "frozen" in the fluid motion. A passively evolving vector field $\mathbf{V}$ is governed by an equation of the form

$$\partial_t \mathbf{V} + \mathbf{u} \cdot \nabla \mathbf{V} - \mathbf{V} \cdot \nabla \mathbf{u} = 0, \qquad \text{(G.15)}$$

where $\mathbf{u}(x, t)$ represents the velocity field of the fluid. The strength of the vector field can grow or decay during its time evolution. The amplification of the vector field in such a process is called the "dynamo effect." In a strongly chaotic fluid motion we can characterize the asymptotic behavior of the field with an exponent

$$\mathbf{V}(x, t) \sim \mathbf{V}(x)e^{\nu t}, \qquad \text{(G.16)}$$

where ν is called the fast dynamo rate. The goal of this section is to show that periodic orbit theory can be developed for such a highly non-trivial system as well.

We can write the solution of (G.15) formally, as shown by Cauchy. Let $\mathbf{x}(t, \mathbf{a})$ be the position of the fluid particle that was at the point $\mathbf{a}$ at $t = 0$. Then the field evolves according to

$$\mathbf{V}(\mathbf{x}, t) = \mathbf{J}(\mathbf{a}, t)\mathbf{V}(\mathbf{a}, 0) \quad , \qquad \text{(G.17)}$$

where $\mathbf{J}(\mathbf{a}, t) = \partial(\mathbf{x})/\partial(\mathbf{a})$ is the fundamental matrix of the transformation that moves the fluid into itself $\mathbf{x} = \mathbf{x}(\mathbf{a}, t)$.

We write $\mathbf{x} = f^t(\mathbf{a})$, where f^t is the flow that maps the initial positions of the fluid particles into their positions at time t. Its inverse, $\mathbf{a} = f^{-t}(\mathbf{x})$, maps particles at time t and position $\mathbf{x}$ back to their initial positions. Then we can write (G.17)

$$V_i(\mathbf{x}, t) = \sum_j \int d^3\mathbf{a}\, \mathcal{L}^t_{ij}(\mathbf{x}, \mathbf{a}) V_j(\mathbf{a}, 0) \quad , \qquad \text{(G.18)}$$

with

$$\mathcal{L}^t_{ij}(\mathbf{x}, \mathbf{a}) = \delta(\mathbf{a} - f^{-t}(\mathbf{x})) \frac{\partial x_i}{\partial a_j} \quad . \qquad \text{(G.19)}$$

For large times, the effect of $\mathcal{L}^t$ is dominated by its leading eigenvalue, $e^{\nu_0 t}$ with $Re(\nu_0) > Re(\nu_i)$, $i = 1, 2, 3, ...$ In this way the transfer operator furnishes the fast dynamo rate, $\nu := \nu_0$.

The trace of the transfer operator is the sum over all periodic orbit contributions, with each cycle weighted by its intrinsic stability

$$\mathbf{Tr}\mathcal{L}^t = \sum_p T_p \sum_{r=1}^{\infty} \frac{\operatorname{tr} M_p^r}{\left|\det\left(\mathbf{1} - M_p^{-r}\right)\right|} \delta(t - rT_p). \qquad \text{(G.20)}$$

We can construct the corresponding spectral determinant as usual

$$F(s) = \exp\left[-\sum_p \sum_{r=1}^{\infty} \frac{1}{r} \frac{\operatorname{tr} M_p^r}{\left|\det\left(\mathbf{1} - M_p^{-r}\right)\right|} e^{srT_p} \right] \quad . \qquad \text{(G.21)}$$

Note that in this formuli we have omitted a term arising from the Jacobian transformation along the orbit which would give $1 + \mathrm{tr}\ M_p^r$ in the numerator rather than just the trace of M_p^r. Since the extra term corresponds to advection along the orbit, and this does not evolve the magnetic field, we have chosen to ignore it. It is also interesting to note that the negative powers of the Jacobian occur in the denominator, since we have f^{-t} in (G.19).

In order to simplify $F(s)$, we factor the denominator cycle stability determinants into products of expanding and contracting eigenvalues. For a 3-dimensional fluid flow with cycles possessing one expanding eigenvalue Λ_p (with $|\Lambda_p| > 1$), and one contracting eigenvalue λ_p (with $|\lambda_p| < 1$) the determinant may be expanded as follows:

$$\left|\det\left(\mathbf{1} - M_p^{-r}\right)\right|^{-1} = |(1-\Lambda_p^{-r})(1-\lambda_p^{-r})|^{-1} = |\lambda_p|^r \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \Lambda_p^{-jr}\lambda_p^{kr} \quad . \tag{G.22}$$

With this decomposition we can rewrite the exponent in (G.21) as

$$\sum_p \sum_{r=1}^{\infty} \frac{1}{r} \frac{(\lambda_p^r + \Lambda_p^r)e^{srT_p}}{\left|\det\left(\mathbf{1} - M_p^{-r}\right)\right|} = \sum_p \sum_{j,k=0}^{\infty}\sum_{r=1}^{\infty} \frac{1}{r}\left(|\lambda_p|\Lambda_p^{-j}\lambda_p^k e^{sT_p}\right)^r (\lambda_p^r + \Lambda_p^r) \quad , \tag{G.23}$$

which has the form of the expansion of a logarithm:

$$\sum_p \sum_{j,k} \left[\log\left(1 - e^{sT_p}|\lambda_p|\Lambda_p^{1-j}\lambda_p^k\right) + \log\left(1 - e^{sT_p}|\lambda_p|\Lambda_p^{-j}\lambda_p^{1+k}\right)\right] \quad . \tag{G.24}$$

The spectral determinant is therefore of the form,

$$F(s) = F_e(s)F_c(s) \quad , \tag{G.25}$$

where

$$F_e(s) = \prod_p \prod_{j,k=0}^{\infty} \left(1 - t_p^{(jk)}\Lambda_p\right) \quad , \tag{G.26}$$

$$F_c(s) = \prod_p \prod_{j,k=0}^{\infty} \left(1 - t_p^{(jk)}\lambda_p\right), \tag{G.27}$$

with

$$t_p^{(jk)} = e^{sT_p}|\lambda_p| \frac{\lambda_p^k}{\Lambda_p^j} \quad . \tag{G.28}$$

The two factors present in $F(s)$ correspond to the expanding and contracting exponents. (Had we not neglected a term in (G.21), there would be a third factor corresponding to the translation.)

For 2-d Hamiltonian volume preserving systems, $\lambda = 1/\Lambda$ and (G.26) reduces to

$$F_e(s) = \prod_p \prod_{k=0}^{\infty} \left(1 - \frac{t_p}{\Lambda_p^{k-1}}\right)^{k+1}, \quad t_p = \frac{e^{sT_p}}{|\,\Lambda_p\,|} \quad . \tag{G.29}$$

With $\sigma_p = \Lambda_p/|\Lambda_p|$, the Hamiltonian zeta function (the $j = k = 0$ part of the product (G.27)) is given by

$$1/\zeta_{dyn}(s) = \prod_p \left(1 - \sigma_p e^{sT_p}\right). \tag{G.30}$$

This is a curious formula — the zeta function depends only on the return times, not on the eigenvalues of the cycles. Furthermore, the identity,

$$\frac{\Lambda + 1/\Lambda}{|(1-\Lambda)(1-1/\Lambda)|} = \sigma + \frac{2}{|(1-\Lambda)(1-1/\Lambda)|} \quad ,$$

when substituted into (G.25), leads to a relation between the vector and scalar advection spectral determinants:

$$F_{dyn}(s) = F_0^2(s)/\zeta_{dyn}(s). \tag{G.31}$$

The spectral determinants in this equation are entire for hyperbolic (axiom A) systems, since both of them correspond to multiplicative operators.

In the case of a flow governed by a map, we can adapt the formulas (G.29) and (G.30) for the dynamo determinants by simply making the substitution

$$z^{n_p} = e^{sT_p} \quad , \tag{G.32}$$

where n_p is the integer order of the cycle. Then we find the spectral determinant $F_e(z)$ given by equation (G.29) but with

$$t_p = \frac{z^{n_p}}{|\Lambda_p|} \tag{G.33}$$

for the weights, and

$$1/\zeta_{dyn}(z) = \Pi_p\left(1 - \sigma_p z^{n_p}\right) \tag{G.34}$$

for the zeta-function

For *maps* with finite Markov partition the inverse zeta function (G.34) reduces to a polynomial for z since curvature terms in the cycle expansion vanish. For example, for maps with complete binary partition, and with the fixed point stabilities of opposite signs, the cycle expansion reduces to

$$1/\zeta_{dyn}(s) = 1. \tag{G.35}$$

For such *maps* the dynamo spectral determinant is simply the square of the scalar advection spectral determinant, and therefore all its zeros are double. In other words, for flows governed by such discrete maps, the fast dynamo rate equals the scalar advection rate.

In contrast, for 3-dimensional *flows*, the dynamo effect is distinct from the scalar advection. For example, for flows with finite symbolic dynamical grammars, (G.31) implies that the dynamo zeta function is a ratio of two entire determinants:

$$1/\zeta_{dyn}(s) = F_{dyn}(s)/F_0^2(s)\,. \tag{G.36}$$

This relation implies that for *flows* the zeta function has double poles at the zeros of the scalar advection spectral determinant, with zeros of the dynamo spectral determinant no longer coinciding with the zeros of the scalar advection spectral determinant; Usually the leading zero of the dynamo spectral determinant is larger than the scalar advection rate, and the rate of decay of the magnetic field is no longer governed by the scalar advection. [exercise G.2]

Commentary

Remark G.1 Dynamo zeta. The dynamo zeta (G.34) has been introduced by Aurell and Gilbert [2] and reviewed in ref. [3]. Our exposition follows ref. [19].

Exercises

G.1. **Stretching factor.** Prove the multiplicative property of the stretching factor (G.2). Why should we extend the phase space with the tangent space?

G.2. **Dynamo rate.** Suppose that the fluid dynamics is highly dissipative and can be well approximated by the piecewise linear map

$$f(x) = \begin{cases} 1 + ax & \text{if} \quad x < 0, \\ 1 - bx & \text{if} \quad x > 0, \end{cases} \tag{G.37}$$

on an appropriate surface of section ($a, b > 2$). Suppose also that the return time is constant T_a for $x < 0$ and T_b