A MODEL OF VELOCITY DEPENDENCE OF ATOMIC FRICTION

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Abstract

A model is to consider a cantilever tip of an Atomic Force Microscope(AFM) as a bouncing ball, which attached with two springs in x and y direction. Poincare section is chosen to be the periodical surface of the sample

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I. INTRODUCTION

A. A Model with Sinusoidal Surface

The dynamical equations of trajectory:

$$\begin{aligned}
\dot{x_1} &= -k_l(x_3 - Vt) \\
\dot{x_2} &= -k_n(x_4 - x_{40}) \\
\dot{x_3} &= x_1/m \\
\dot{x_4} &= x_2/m
\end{aligned}$$

where the variables are

$$\begin{aligned}
x_1 &\rightarrow \textit{horizontal momentum } p_x \\
x_2 &\rightarrow \textit{vertical momentum } p_y \\
x_3 &\rightarrow \textit{horizontal position } x \\
x_4 &\rightarrow \textit{vertical position } y
\end{aligned}$$

The spring constants are $k_l = 66.7N/m$ and $k_n = 0.12N/m$. The equivalent mass of the tip is set to be $8.6 \times 10^{-9}kg$, which is calculated from the frequency of free vibration of the cantilever. The spring is dragged in a constant speed V in the x direction. The equilibrium point in y direction is set to be 0. The dynamical equations are changed to

$$\begin{aligned}
\dot{x_1} &= -66.7\frac{N}{m}(x_3 - vt) \\
\dot{x_2} &= -0.12x_4 \\
\dot{x_3} &= 1.16 \times 10^{-9}kg^{-1}x_1 \\
\dot{x_4} &= 1.16 \times 10^{-9}kg^{-1}x_2
\end{aligned}$$

Choose the dimension to be

$$[L] = nm$$

$$[M] = \mu g$$

$$[T] = ms$$

Then the dynamical euqations are

$$
\begin{aligned}
\dot{x_1} &= -66700(x_3 - Vt) \\
\dot{x_2} &= -120x_4 \\
\dot{x_3} &= 1.16x_1 \\
\dot{x_4} &= 1.16x_2
\end{aligned}
$$

After modified the parameters, the equations are

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x} + b(t)$$

where

$$A(t) = \begin{bmatrix} 0 & 0 & -66700 & 0 \\ 0 & 0 & 0 & -120 \\ 1.16 & 0 & 0 & 0 \\ 0 & 1.16 & 0 & 0 \end{bmatrix},$$

$$b(t) = \begin{bmatrix} Vt \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The eigenvalues of A

$$\begin{bmatrix} ai \\ -ai \\ bi \\ -bi \end{bmatrix}$$

where

$$a = 278.158228352138, \quad b = 11.798304963002$$

The matrix of eigenvectors is

$$\begin{bmatrix} -ci & ci & 0 & 0 \\ 0 & 0 & -f & -f \\ -d & -d & 0 & 0 \\ 0 & 0 & ei & -ei \end{bmatrix}$$

where

$$
\begin{aligned}
c &= 0.999991304461 \\
d &= 0.004170252018 \\
e &= 0.995201428420 \\
f &= 0.97847416268
\end{aligned}
$$

The solution of these equations with initial condition (x_0, t_0) is

$$\mathbf{x}(t) = e^{A(t-t_0)}\mathbf{x}_0 + \int_{t_0}^{t} e^{A(t-\tau)} b(\tau) d\tau$$

The first term turns out to be

$$\begin{bmatrix} x_{01}\cos a(t-t_0) - jx_{03}\sin a(t-t_0) \\ x_{02}\cos b(t-t_0) - kx_{04}\sin b(t-t_0) \\ x_{01}g\cos a(t-t_0) + x_{03}\sin a(t-t_0) \\ x_{02}h\cos b(t-t_0) + x_{04}\sin b(t-t_0) \end{bmatrix}$$

The second term

$$\begin{bmatrix} n + (qt_0 - p)\sin a(t-t_0) - n\cos a(t-t0) \\ -r\sin b(t-t_0) \\ ut - s - (ut_0 - s)\cos a(t-t_0) - v\sin a(t-t_0) \\ -w + w\cos b(t-t_0) \end{bmatrix}$$

where the constants are

$$\begin{aligned}
g &= -d/c = 0.04170288281 \\
h &= -f/e = 0.098319208025 \\
j &= -c/d = 239.791576165636 \\
k &= -e/f = 10.170952554312 \\
n &= k_l V/a^2 \\
q &= k_l V/a \\
r &= 0 \\
u &= g k_l V/a \\
v &= g k_l V/a^2 \\
w &= 0
\end{aligned}$$

So the solution is

$$\mathbf{x}(t) = \begin{bmatrix} n + (qt_0 - p - jx_{03}) \sin a(t - t_0) + (x_{01} - n) \cos a(t - t0) \\ x_{02} \cos b(t - t_0) - (kx_{04}) \sin b(t - t_0) \\ ut - s - (ut_0 - s - x_{03}) \cos a(t - t_0) + (gx_{01} - v) \sin a(t - t_0) \\ x_{02} h \sin b(t - t_0) + x_{04} \cos b(t - t_0) \end{bmatrix}$$

When the cantilever bounced back from the sample surface, the velocities are changed. Due to the coefficent of restitution, there is also lost of momentum. The new velocities can be calculated in the following way:

$$\mathbf{v}' = T^{-1} R T \mathbf{v}$$

where T is the transformation matrix of coordinates.

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

with θ that is the tilted angle of the sample surface. R is the bouncing matrix

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix}$$

with α that is the coefficent of restitution.

The sample surface has set to be a one dimensional sinusoidal function along the moving direction of the cantilever.

$$x_4 = 0.375 + 0.025 \sin(\frac{2\pi(x_3 - 0.1)}{0.4})$$

The slope of the surface at where the tip bounced is

$$F = \frac{dx_4}{dx_3} = 0.125\pi \cos(\frac{2\pi(x_3 - 0.1)}{0.4})$$

The angle of tilted surface is

$$\theta = arctan(F)$$

The matrix of bouncing can be found as

$$T^{-1}RT = \frac{1}{1+F^2}\begin{pmatrix} 1-\alpha F^2 & (1+\alpha)F \\ (1+\alpha)F & -\alpha + F^2 \end{pmatrix}$$

So the solution of the system after bouncing will be

$$\mathbf{x_f}(t) = \begin{pmatrix} \frac{1}{1+F^2}\begin{pmatrix} 1-\alpha F^2 & (1+\alpha)F \\ (1+\alpha)F & -\alpha + F^2 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$
$$\begin{pmatrix} n + (qt_0 - p - jx_{03}) \sin a(t-t_0) + (x_{01} - n) \cos a(t - t0) \\ x_{02} \cos b(t-t_0) - (kx_{04}) \sin b(t-t_0) \\ ut - s - (ut_0 - s - x_{03}) \cos a(t-t_0) + (gx_{01} - v) \sin a(t-t_0) \\ x_{02} h \sin b(t-t_0) + x_{04} \cos b(t-t_0) \end{pmatrix}$$

B. Map on the Poincare Sectoin

Choose the surface as the Poincare sectoin. The three variables on the Poincare section are momentums in two directoins ζ, η and the horizontal position ϕ.

$$\begin{aligned} \zeta &= \frac{v_x}{V} = \frac{x_1}{mV} \\ \eta &= \frac{v_y}{V} = \frac{x_2}{mV} \\ \phi &= \frac{2\pi(x_3 - 0.1)}{0.4} \quad mod \ 2\pi \end{aligned}$$

Before calculating these variables, the time at when the bouncing happen should be solved. The space envolution equations are

$$\begin{aligned} x_3 &= ut - s - (ut_0 - s - x_{03}) \cos a(t - t_0) + (gx_{01} - v) \sin a(t - t_0) \\ x_4 &= x_{02} h \sin b(t - t_0) + x_{04} \cos b(t - t_0) \end{aligned}$$

On the surface we have

$$x_4 = 0.375 + 0.025 \sin(\frac{2\pi(x_3 - 0.1)}{0.4})$$

So the time t can be solved from

$$\begin{aligned} & x_{02} h \sin b(t - t_0) + x_{04} \cos b(t - t_0) \\ = \ & 0.375 + 0.025 \sin(5\pi[ut - s - (ut_0 - s - x_{03}) \cos a(t - t_0) + (gx_{01} - v) \sin a(t - t_0)]) \end{aligned}$$

Although the explicit expression of t cannot be solved, we can solve it numerically in matlab(using function fzero.m). The momentums before bouncing are

$$\begin{aligned} \zeta' &= (n + (qt_0 - p - j(\phi_0/5\pi + 0.1)) \sin a(t - t_0) + (\zeta_0 mV - n) \cos a(t - t0))/mV \\ \eta' &= (\eta_0 mV \cos b(t - t_0) - (k0.375 + 0.025 \sin(\phi_0)) \sin b(t - t_0))/mV \end{aligned}$$

After bouncing they change to be

$$\begin{pmatrix} \zeta \\ \eta \end{pmatrix} = \frac{1}{1 + F^2} \begin{pmatrix} 1 - \alpha F^2 & (1 + \alpha)F \\ (1 + \alpha)F & -\alpha + F^2 \end{pmatrix} \begin{pmatrix} \zeta' \\ \eta' \end{pmatrix} = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_3 \end{pmatrix} \begin{pmatrix} \zeta' \\ \eta' \end{pmatrix}$$

where

$$\begin{aligned} F &= 0.125\pi \cos\phi \\ M_1 &= \frac{1 - \alpha F^2}{1 + F^2} \\ M_2 &= \frac{(1 + \alpha)F}{1 + F^2} \\ M_3 &= \frac{-\alpha + F^2}{1 + F^2} \end{aligned}$$

and the position is

$$\phi = ut - s - (ut_0 - s - \phi_0/5\pi + 0.1) \cos a(t - t_0) + (g\zeta_0 mV - v) \sin a(t - t_0)$$

The Jacobian of the map before bouncing is

$$J' = \begin{pmatrix} \frac{\partial\zeta'}{\partial\zeta_0} & \frac{\partial\zeta'}{\partial\eta_0} & \frac{\partial\zeta'}{\partial\phi_0} \\ \frac{\partial\eta'}{\partial\zeta_0} & \frac{\partial\eta'}{\partial\eta_0} & \frac{\partial\eta'}{\partial\phi_0} \\ \frac{\partial\phi}{\partial\zeta_0} & \frac{\partial\phi}{\partial\eta_0} & \frac{\partial\phi}{\partial\phi_0} \end{pmatrix} = \begin{pmatrix} \cos a(t-t_0) & 0 & -\frac{j}{5\pi}\sin a(t-t_0) \\ 0 & \cos b(t-t_0) & 0 \\ mVg\sin a(t-t_0) & 0 & \frac{1}{5\pi}\cos a(t-t_0) \end{pmatrix}$$

The Jacobian of the map after bounce is

$$J = \begin{pmatrix} \frac{\partial\zeta}{\partial\zeta_0} & \frac{\partial\zeta}{\partial\eta_0} & \frac{\partial\zeta}{\partial\phi_0} \\ \frac{\partial\eta}{\partial\zeta_0} & \frac{\partial\eta}{\partial\eta_0} & \frac{\partial\eta}{\partial\phi_0} \\ \frac{\partial\phi}{\partial\zeta_0} & \frac{\partial\phi}{\partial\eta_0} & \frac{\partial\phi}{\partial\phi_0} \end{pmatrix}$$

Where

$$\begin{aligned}
\frac{\partial\zeta}{\partial\zeta_0} &= \frac{\partial M_1}{\partial F}\frac{\partial F}{\partial\phi}\frac{\partial\phi}{\partial\zeta_0}\zeta' + M_1\frac{\partial\zeta'}{\partial\zeta_0} + \frac{\partial M_2}{\partial F}\frac{\partial F}{\partial\phi}\frac{\partial\phi}{\partial\zeta_0}\eta' \\
\frac{\partial\zeta}{\partial\eta_0} &= M_2\frac{\partial\eta'}{\partial\eta_0} \\
\frac{\partial\zeta}{\partial\phi_0} &= \frac{\partial M_1}{\partial F}\frac{\partial F}{\partial\phi}\frac{\partial\phi}{\partial\phi_0}\zeta' + M_1\frac{\partial\zeta'}{\partial\phi_0} + \frac{\partial M_2}{\partial F}\frac{\partial F}{\partial\phi}\frac{\partial\phi}{\partial\phi_0}\eta' \\
\frac{\partial\eta}{\partial\zeta_0} &= \frac{\partial M_2}{\partial F}\frac{\partial F}{\partial\phi}\frac{\partial\phi}{\partial\zeta_0}\zeta' + M_2\frac{\partial\zeta'}{\partial\zeta_0} + \frac{\partial M_3}{\partial F}\frac{\partial F}{\partial\phi}\frac{\partial\phi}{\partial\zeta_0}\eta' \\
\frac{\partial\eta}{\partial\eta_0} &= M_3\frac{\partial\eta'}{\partial\eta_0} \\
\frac{\partial\eta}{\partial\phi_0} &= \frac{\partial M_2}{\partial F}\frac{\partial F}{\partial\phi}\frac{\partial\phi}{\partial\phi_0}\zeta' + M_2\frac{\partial\zeta'}{\partial\phi_0} + \frac{\partial M_3}{\partial F}\frac{\partial F}{\partial\phi}\frac{\partial\phi}{\partial\phi_0}\eta' \\
\frac{\partial M_1}{\partial F} &= -\frac{2F\alpha}{1+F^2} - \frac{2F(1-F^2\alpha)}{(1+F^2)^2} \\
\frac{\partial M_2}{\partial F} &= -\frac{2F^2(1+\alpha)}{(1+F^2)^2} + \frac{1+\alpha}{1+F^2} \\
\frac{\partial M_3}{\partial F} &= \frac{2F}{1+F^2} - \frac{2F(F^2-\alpha)}{(1+F^2)^2} \\
\frac{\partial F}{\partial\phi} &= -0.125\pi\sin\phi
\end{aligned}$$

This model has two frequencies, a and b. If the ratio of the two frequencies is not rational number, for elastic bouncing, there will be no stable periodical solution. However, since the system is dissipative, they could exist. This is shown in the old model, which has a different sample surface form with circular scatters. A figure has been obtained by numerical assimilation to show the stable periodical solution can exist between the chaotical areas, which known as Arnold tongues.

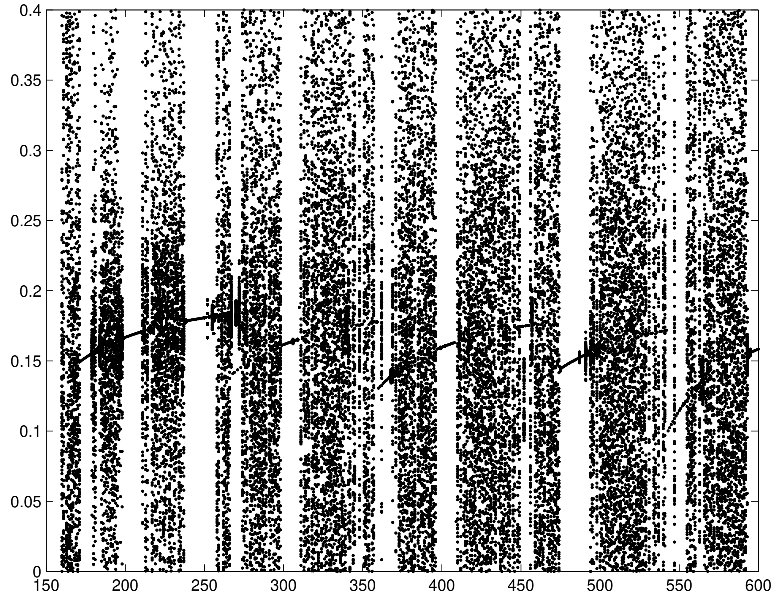


FIG. 1: v_y versus kv_x. kv_x is from 160 to 600 thousands, coressponding to v_x from 2.39 to 8.99

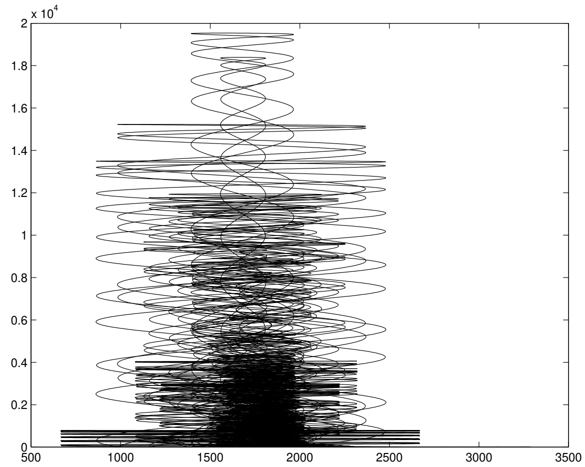
Clearly stable solutions depend on horizontal velocities. In those areas outside Arnold tongues, there can only exist one solution of period one. Even within the Arnold tongues, one cannot be sure that it is chaotic because our numerical assimilation is short for each horizontal velocities. Those Arnold tongues would be thinner if the numerical calculation is ran longer, but it can be never long enough.

When same technique has been put on the current model, two conditions could happen and the calculation could not ran longer. The amplitude could be either too big or too small as shown in figures below. For huge amplitude of bouncing, with the same step size, the bouncing point found could be inaccurate. Tiny amplitude just means sliding. No interested dynamics at all.

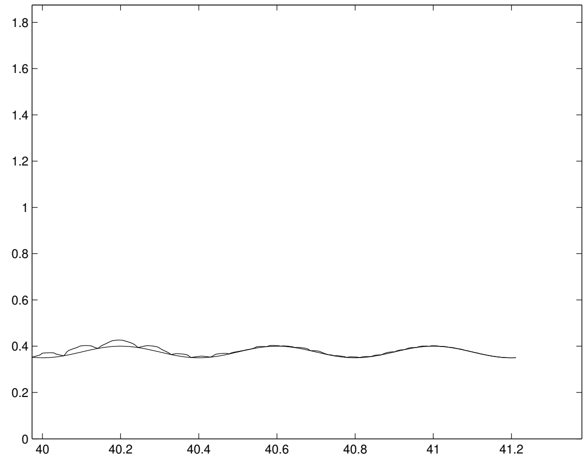
Play with the parameters could be a tedious job but one chaotic trajectory has been found. This trajectory has no huge amplitude and is not going to die out in a short time. It is obtained after a number huge bounce with a large horizontal velocity(~ 24).

No stable periodic solution has been found yet.

Another problem is that the Jacobian is still a function of time, which is always evolving while the variables are mapping into the same space.



(a)



(b)

FIG. 2: The amplitude of the trajectory could be huge or too small.

II. DYNAMICAL FRICTION FORMULA

There are two ways to define the friction, the energy lost over distance or the momentum lost over time. The former case is discussed here.

Suppose we have a prime cycle with period n_p and time T_p. The energy lost over one bounce is

$$a_p = K_{before} - K_{after} = \frac{1}{2}m(v_x'^2 + v_y'^2) - \frac{1}{2}m(v_x^2 + v_y^2)$$

where the velocities with prime are before the bounce and those without prime are after. The total lost of energy can be written as

$$A_p = \sum_{i=1}^{n_p} \frac{1}{2}m(v_{xi}'^2 + v_{yi}'^2) - (v_{xi}^2 + v_{yi}^2) = \sum_{i=1}^{n_p} \frac{mV}{2}[(\zeta_i'^2 + \eta_i'^2) - (\zeta_i^2 + \eta_i^2)]$$

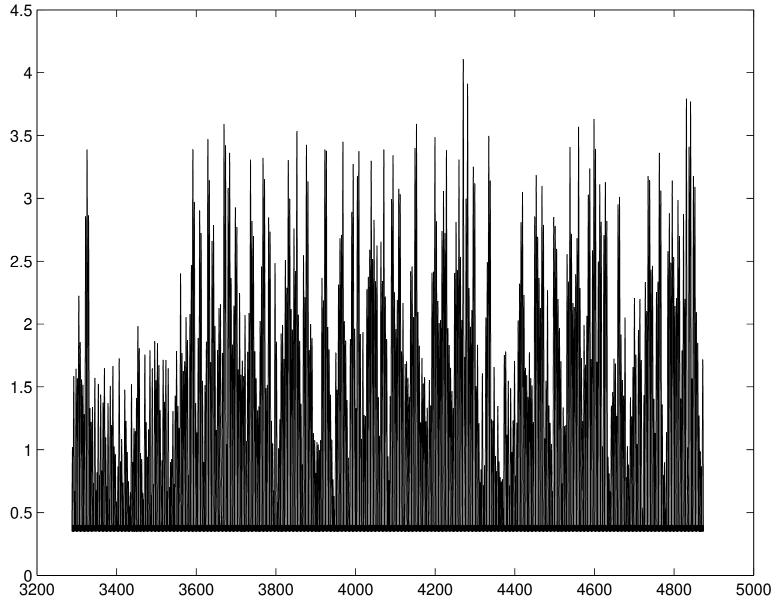


FIG. 3: One chaotic trajectory.

The special determinant of an evolution operator for the map(from formula 8.3 of chapter 8 of the chaos book.)

$$det(1-zL)=exp\left(-\sum_{p}\sum_{r=1}^{\infty}\frac{1}{r}\frac{z^{n_p r}e^{r\beta A_p}}{|det(1-J_p^r)|}\right)$$

where L is the evolution operator and we have

$$< e^{\beta At} >=< L^t >$$

Since we have the Jacobian of one bounce, the Jacobian of one cycle of length n_p is

$$J_p=\sum_{i=1}^{n_p}J_i$$

For enough prime cycles and enough repeat times r, the special determinant can give us the eigenvalues of evolution operator L, in which the biggest solution z_0(in fact $1/L$ here) will dominant. The limit of growth rate $s(\beta)$ can be found by

$$s(\beta)=-\ln z_0$$

The expected value of lost of energy can be calculated by taking derivative of $s(\beta)$.

$$< A_p >=\frac{\partial s}{\partial\beta}|_{\beta=0}$$

The friction can be written as the lost of energy over distance

$$f = \frac{< A_p >}{T_p V}$$

III. A TOY MODEL

Besides the model with two springs, a toy model is set up for a rough approach. In this toy model, the horizontal spring is set to have infinite large spring constant. The trajectory will be a sinusoidal function. The system gains or losses momentum immediately while in the previous model the system it takes time.

Another change is made to calculate the time between two bounces. The time of going up after last bounce is assumed to be same as the time of falling down before next bounce. The vertical velocity before next bounce will have the same magnitude as the vertical velocity after last bounce.

The dynamical equations are

$$\begin{aligned} v_x &= const \\ v_y &= A\omega \cos \omega t \\ x &= x_0 + v_x(t - t_0) \\ y &= A \sin \omega t \end{aligned}$$

The initial conditions are

$$\begin{aligned} v_{x0} &= v_x \\ v_{y0} &= A\omega \cos \omega t_0 \\ x_0 &= x_0 \\ y_0 &= A \sin \omega t_0 \end{aligned}$$

So

$$t_0 = \frac{1}{\omega} \arctan(\frac{y_0 \omega}{v_{y0}})$$

The t can be solved since $t - t_0$ is 2 times of the time of going up.

$$t = t_0 + \frac{\pi}{\omega} - \frac{2}{\omega} \arctan(\frac{y_0 \omega}{v_{y0}})$$

The equations can be written as

$$\begin{aligned}
v_x &= v_x \\
v_y &= v_{y0}\cos\omega(t-t_0) - y_0\omega\sin\omega(t-t_0) \\
x &= x_0 + \frac{v_x\pi}{\omega} - \frac{2v_x}{\omega}\arctan(\frac{y_0\omega}{v_{y0}}) \\
y &= \frac{v_{y0}}{\omega}\sin\omega(t-t_0) + y0\cos\omega(t-t_0)
\end{aligned}$$

The surface function is sinusoidal

$$y = 0.375 + 0.025\sin(\frac{2\pi(x-0.1)}{0.4})$$

After bounce the vertical velocity will change while the horizontal velocity stays the same.

$$\begin{aligned}
v'_x &= v_x \\
v'_y &= \frac{F(1+\alpha)v_x - v_{y0}(F^2-\alpha)}{1+F^2}
\end{aligned}$$

where v_y has been replaced by $-v_{y0}$ and F is

$$F = 0.025\cos 5\pi(x-0.1)$$

Let the surface be the Poincare section and set the two variables of map as

$$\zeta = \frac{v_y}{v_x} \quad \phi = 5\pi(x-0.1)$$

The map can be written as

$$\begin{aligned}
\phi &= \phi_0 + \frac{5\pi v_x}{\omega}(\pi - 2\arctan(\rho)) \\
\zeta &= \frac{F(1+\alpha) - \zeta_0(F^2-\alpha)}{1+F^2}
\end{aligned}$$

where

$$\begin{aligned}
\rho &= \frac{y_0\omega}{\zeta_0 v_x} = \frac{(0.375+0.025\sin\phi_0)\omega}{\zeta_0 v_x} \\
F &= 0.025\cos(5\pi(\frac{\rho}{5\pi} + \frac{v_x}{\omega}(\pi - 2\arctan(\rho))))
\end{aligned}$$

Below is a bifurcation diagram of ζ versus changing parameter v_x.

The disadvange of this model is that when the bounce is small, the time difference between going up and falling down can no more be ignored. Also, when vertical velocity is negative after bounce, the mapping could be wrong.

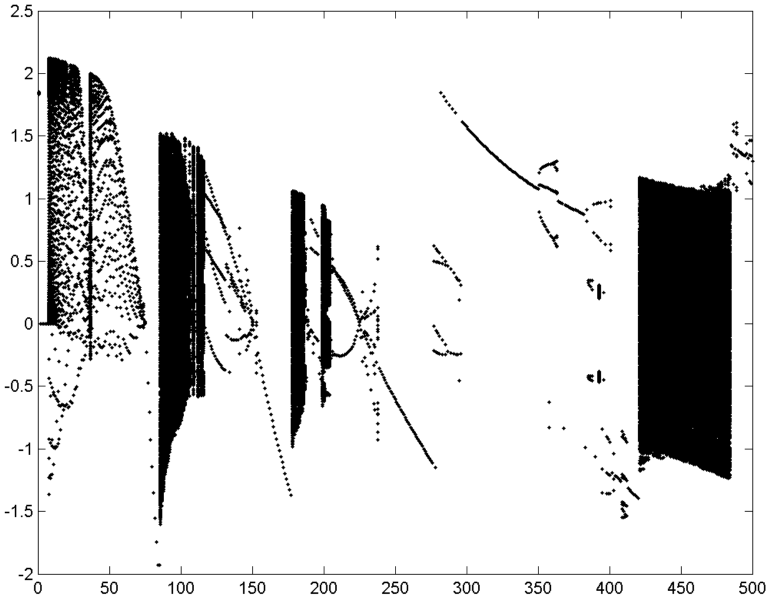


FIG. 4: ζ versus v_x. v_x is from 0 to 5.

IV. ANOTHER MODEL

This is a adjusted toy model in which exact bouncing position is calculated and the spring constant of the horizontal spring is also infinite.
The dynamical equations are

$$\begin{aligned} v_x &= v_x \\ v_y &= v_{y0}\cos\omega(t-t_0) - y_0\omega\sin\omega(t-t_0) \\ x &= x_0 + v_x(t-t_0) \\ y &= \frac{v_{y0}}{\omega}\sin\omega(t-t_0) + y0\cos\omega(t-t_0) \end{aligned}$$

The surface function is

$$y = 0.375 + 0.025\sin(\frac{2\pi(x-0.1)}{0.4})$$

The exact bouncing position x can be solved from

$$\begin{aligned} &0.375 + 0.025\sin(\frac{2\pi(x-0.1)}{0.4}) \\ = &\ \frac{v_{y0}}{\omega}\sin\omega(\frac{x-x_0}{v_x}) + y0\cos\omega(\frac{x-x_0}{v_x}) \end{aligned}$$

Again this is solved numerically by matlab(using function fzero.m). The acurracy normally could be equal or higher than 10^{-14}. Then we can get the velocity just before bounce

$$v_y \;=\; v_{y0}\cos\omega(\frac{x-x_0}{v_x}) - y_0\omega\sin\omega(\frac{x-x_0}{v_x})$$

Let the surface be the Poincare section and set the two variables of map as

$$\zeta = \frac{v_y}{v_x} \quad \phi = 5\pi(x-0.1)$$

The map of these two variable just before bounce can be written as

$$\begin{aligned}
\phi &= \phi_0 + 5\pi v_x(t-t_0) \\
\zeta' &= \zeta_0\cos\omega(t-t_0) - \frac{0.025\omega}{v_x}\sin\omega(t-t_0)\sin\phi_0 - \frac{0.375\omega}{v_x}\sin\omega(t-t_0)
\end{aligned}$$

After bouce, there is no change for ϕ and ζ changes to

$$\zeta = \frac{F(1+\alpha) - \zeta'(F^2-\alpha)}{1+F^2}$$

where

$$F = 0.025\cos\phi$$

We make further change of the variables and let it map into the fundamental domain and single value.

$$\sigma = \cos\frac{\phi}{2}$$

The map of the new variables can be written as

$$\begin{aligned}
\sigma &= \sigma_0\cos(2.5\pi v_x(t-t_0)) - (1-\sigma_0^2)^{\frac{1}{2}}\sin(2.5\pi v_x(t-t_0)) \\
\zeta &= M_1\zeta' + M_2
\end{aligned}$$

where

$$\begin{aligned}
M_1 &= \frac{F^2-\alpha}{1+F^2} \\
M_2 &= \frac{1+\alpha}{1+F^2} \\
F &= 0.125\pi(2\sigma^2-1) \\
\zeta' &= \zeta_0\cos\omega(t-t_0) - \frac{0.025\omega}{v_x}\sin\omega(t-t_0)\sin\phi_0 - \frac{0.375\omega}{v_x}\sin\omega(t-t_0)
\end{aligned}$$

The jacobian of this map is

$$J = \begin{pmatrix} \frac{\partial\zeta}{\partial\zeta_0} & \frac{\partial\zeta}{\partial\sigma_0} \\ \frac{\partial\sigma}{\partial\zeta_0} & \frac{\partial\sigma}{\partial\sigma_0} \end{pmatrix}$$

Where

$$\begin{aligned}
\frac{\partial\zeta}{\partial\zeta_0} &= M_1\frac{\partial\zeta'}{\partial\zeta_0} \\
\frac{\partial\zeta}{\partial\sigma_0} &= \frac{\partial M_1}{\partial F}\frac{\partial F}{\partial\sigma}\frac{\partial\sigma}{\partial\sigma_0}\zeta' + M_1\frac{\partial\zeta'}{\partial\sigma_0} + \frac{\partial M_2}{\partial F}\frac{\partial F}{\partial\sigma}\frac{\partial\sigma}{\partial\sigma_0} \\
\frac{\partial M_1}{\partial F} &= \frac{2F}{1+F^2} - \frac{2F(F^2-\alpha)}{(1+F^2)^2} \\
\frac{\partial M_2}{\partial F} &= -\frac{2F^2(1+\alpha)}{(1+F^2)^2} + \frac{1+\alpha}{1+F^2} \\
\frac{\partial F}{\partial\sigma} &= 0.5\pi\sigma \\
\frac{\partial\sigma}{\partial\zeta_0} &= 0 \\
\frac{\partial\sigma}{\partial\sigma_0} &= \cos(2.5\pi v_x(t-t_0)) + \frac{1}{(1-\sigma_0^2)^{\frac{1}{2}}}\sin(2.5\pi v_x(t-t_0))
\end{aligned}$$

A. Problem

From previous results one can assume that there could exist stable periodical solutions with certain v_x. To test these assumptions we evolve the trajectories for sufficently long time over different v_x with random initial conditions. Those trajectories won't be stable after a long time running are likely the interested ones in the chaotic region.
A major problem in doing so is sliding. Since the horizontal velocity is set to be a constant immediately after bounce. The total velocity could point into the surface and the tip would slide on the surface until the tangential direction of the surface is point out of the surface. The vertical velocity then is also fixed. So if slide happened, the infomation before it is lost. The trajectory will change into another one with fixed initial condition. A more careful consideration should be applied for avoiding the slide.

V. DISCUSSION

We have tried four models of friction.
The first model is a bouncing ball connected with two springs in different directions moving

in a constant speed on a surface periodically set up in the shape of circle tops.
We have found that only when speed is high(much larger than experimental value) there would exist periodical solutions. In some region there is a period-one stable solution and in other region it is seems chaotic. We say "seems" because the trajectory is still very wild after a long run. One may never test it long enough to be sure it is chaotic or not.
The Jacobian has been calculated along the trojectory and has been used to find the periodic solution. Several cycle-like solution had been found but after further test they are actually not periodic solution. The speed has not been set right. With that speed, all trajectories will eventually fall on the surface.
Another disadvantage of this model is the surface shape. Obviously the corner between two circle top will cause trouble. Since the surface is in pieces of circle top, difficuties will occur in calculating the bouncing point with big amplitude of bouncing. The second model is the same bouncing ball with a sinusoidal surface. The trajectories are too wild in the same test region of speed as in the last model. The bouncing is easy to have huge amplitude or it could just slide on the surface with numbers of tiny bounce. One may guess that the stable solution may occur with even higher speed since the surface is actually "smoother" than before. A map of variables on the Poincare section that chosen to be on the surface has been set. The Jacobian of this map has been found. One can calculate the Lyapunov exponent or using the Jacobian to get the periodic solution.
The third model is a toy model. The horizontal spring has infinite large spring constant. The time of bouncing up has been set to equal to the time of falling down. A map of variables on the same Poincare section has been set. This model is simpler because it does contain time the Jacobian. It is only good for the trajectories that go up after every bounce. For those go down after bounce, the model will fail.
The last model is a adjusted toy model. With fixed horizontal speed, one have to calculate the bouncing point and the Jacobian calculated is also depend on time. Slide will occur when the trajectory goes "into" the surface after bounce. One may still be able get the periodic solution by carefully trying different initial conditions and parameters. Obviously better idea is need.