Coherent state for the cranked harmonic oscillator in dipole field

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1 Introduction

A cranked anisotropic harmonic oscillator potential was found to lend itself easily to analytical treatments [1]. Glas *et al* (1978) succeeded by using canonical transformation in obtaining an exact solution for the wave functions of this problem in coordinate space[2]. For the one-dimensional harmonic oscillator it is well known that the position-momentum minimum-uncertainty state gives the corresponding classical behaviour for the expectation values of the position and momentum operators and, thus, evolves in time as a coherent state. A cranked harmonic oscillator in dipole field is shown to give the exact classical motion for the original cranked oscillator and to have the same probability density form as the original cranked oscillator.

2 Hamiltonian and Canonical transformation

The Hamiltonian for a particle moving in the dipole fields of a triaxial harmonic oscillator which is rotated around the a axis (we suppose the axis is

1-axis) with constant frequency ω is in the rotating frame of reference given by:

$$H = \sum_{i=1}^{3} \left(\frac{{p_i}^2}{2m} + \frac{1}{2} m \omega_i^2 x_i^2 \right) - \omega l_1 + d_2 x_2 + d_3 x_3 \tag{1}$$

where d_i $(i = 2,3)$ are dipole constant, and $l_1 = x_2 p_3 - x_3 p_2$, which is the componet of its angular momentum operator along the x_1 axis.

It is assumed here that the rotation is taking place about the x_1 axis with constant angular velocity ω. The dipole field is radiating along x_2, x_3 axis. Splitting H into $H = H_1 + H_\omega$ with

$$H_1 = \frac{p_1^2}{2m} + \frac{1}{2} m \omega_1^2 x_1^2 \tag{2}$$

and

$$H_\omega = \sum_{i=2,3} \left(\frac{p_i^2}{2m} + \frac{1}{2} \omega_i^2 x_i^2 \right) - \omega (x_2 p_3 - x_3 P_2) + d_2 x_2 + d_3 x_3 \tag{3}$$

It is evident that the eigenfunctions of H can be factorized into

$$\Psi(r) = \phi(x_1) \psi(x_2, x_3) \tag{4}$$

where $\phi(x_1)$ are the well-known oscillator eigenfuncions of H_1 and $\psi(x_2, x_3)$ are eigenfunctions of H_ω still to be determined.

The eigenvalues of H_ω can be found by canonical transformation of the coordinates x_2, x_3 and their conjugate momenta p_2, and p_3 to new primed coordinates and momenta :

$$\begin{aligned} x_2' &= A x_2 + B p_3 \\ x_3' &= A x_3 + B p_2 \\ p_2' &= p_2 + C x_3 \\ p_3' &= p_3 + C x_2. \end{aligned} \tag{5}$$

Demanding that the primed and unprimed operators are connected by a unitary transformation leads to the requirements[3] that

(a) the commutators between coordinates and momenta are preserved

(b) the primed operators are hermitian.

Requirement (a) is fulfilled for

$$A - BC = 1 \tag{6}$$

Inserting the inverse transformation

$$\begin{aligned} x_2 &= x_2' - Bp_3' \\ x_3 &= x_3' - Bp_2' \\ p_2 &= Ap_2' - Cx_3' \\ p_3 &= Ap_3' - Cx_2' \end{aligned} \tag{7}$$

into the Hamiltonian H_ω and requiring that the transformed operator takes the form of two uncoupled harmonic oscillatiors. This leads to three equations for the three unknown transformation coefficients A, B, and C whose solutions give

$$A = \frac{1}{2}\left[1 + \frac{\omega_3^2 - \omega_2^2}{S}\right], \quad B = \frac{2\omega}{mS}, \quad C = \frac{m}{4\omega}\left(\omega_3^2 - \omega_2^2 - S\right) \tag{8}$$

with

$$S = sign(\omega_3 - \omega_2)\sqrt{\left(\omega_3^2 - \omega_2^2\right)^2 + 8\omega^2\left(\omega_3^2 + \omega_2^2\right)} \tag{9}$$

The new oscillator frequencies and masses are found as

$$\begin{aligned} \Omega_2^2 &= \frac{1}{2}\left(\omega_2^2 + \omega_3^2\right) + \omega^2 - \frac{1}{2}S \\ \Omega_3^2 &= \frac{1}{2}(\omega_2^2 + \omega_3^2) + \omega^2 + \frac{1}{2}S \\ m_2 &= m\frac{\Omega_2^2 - \Omega_3^2}{\Omega_2^2 - \omega_3^2 + \omega^2} \\ m_3 &= m\frac{\Omega_3^2 - \Omega_2^2}{\Omega_3^2 - \omega_2^2 + \omega^2}. \end{aligned} \tag{10}$$

The transformed Hamiltonian in primed coordinates and momenta is

$$H_\omega' = \sum_{i,j=2,3\ i\neq j}\left[\frac{\left(p_i' - m_i B d_j\right)^2}{2m_i} + \frac{1}{2}m_i\Omega_i^2\left(x_2'\right)^2\right] - H_C' \tag{11}$$

where

$$H_C' = \frac{1}{2}m_2 d_3^2 B + \frac{1}{2}m_3 d_2^2 B^2 + \frac{d_2^2}{2m_2\Omega_2^2} + \frac{d_3^2}{2m_3\Omega_3^2} \tag{12}$$

The Hamilitonian shows the minimum position and momentum are shifted from the minimum postion of Hamiltonian of the original cranked oscillator. By redefining the hamiltonian of equation (11), $H' \equiv H_\omega' + H_C'$, more compact form of Hamiltonian can be found as

$$H' = \sum_{i=2,3}\left(\frac{P_i'^2}{2m_i} + \frac{1}{2}m_i\Omega_i^2 X_i'^2\right) \tag{13}$$

where

$$P_i' \equiv p_i' - m_i d_j B, \quad (i,j = 2,3 \quad i \neq j)$$
$$X_i' \equiv x_i' + \frac{d_i}{m_i \Omega_i^2} \quad (i = 2,3) \tag{14}$$

3 Coherent states

Define the coherent states in terms of its stationary occupation number states, parametrized by three complex quantities α_i $(i = 1,2,3)$

$$|\{\alpha_i\}\rangle = \prod_{i=1}^{3} exp\left(-\frac{|\alpha_i|^2}{4b_i^2}\right) \sum_{n_i=0}^{\infty} \left(\frac{\alpha_i}{\sqrt{2}b_i}\right)^{n_i} \frac{1}{\sqrt{n_i!}} e^{-i\Omega_i\left(n_i+\frac{1}{2}\right)t} |n_i\rangle \tag{15}$$

where

$$b_i = \sqrt{\frac{\hbar}{|m_i|\,\Omega_i}} \tag{16}$$

The motion in the x_1 direction will not be dealt with since it is decoupled from the x_2, x_3 motion and is not affected by the rotation. Since the coherent states defined by the (15) are by construction the ordinary coherent states for the new Hamiltonian H', it follows that the standard results for expectation values still hold in the transformed system

$$\begin{aligned} \langle\{\alpha_i\}; t\,|X_i'|\,\{\alpha_i\}; t\rangle &= \alpha_i^R(t) \\ \langle\{\alpha_i\}; t\,|P_i'|\,\{\alpha_i\}; t\rangle &= \frac{\hbar}{b_i^2(t)}\alpha_i^I(t) \quad (i = 2,3) \end{aligned} \tag{17}$$

where

$$\alpha_i(t) = \alpha_i e^{-i\Omega_i t} \tag{18}$$

and the superscripts R and I stand for real and imaginary parts respectively. The expectation values of coordinates and momenta in the non-primed system can be obtained from equations (17) with the help of the inverse transformation obtained from equations (7). The results can be expressed in the compact form

$$\begin{aligned} \langle\{\alpha_i\}; t\,|x_i|\,\{\alpha_i\}; t\rangle &= \alpha_i^R(t) - \frac{\hbar B}{b_j^2}\alpha_j^I(t) - \frac{d_i}{m_i\Omega_i^2} - m_j d_i B^2 \\ \langle\{\alpha_i\}; t\,|p_i|\,\{\alpha_i\}; t\rangle &= \frac{\hbar A}{b_i^2}\alpha_i^I(t) - C\alpha_j^R(t) + \frac{d_j C}{m_j\Omega_j^2} + m_i d_j AB \end{aligned} \tag{19}$$

where $i, j = 2, 3$ $(j \neq i)$.

The classical solutions $x_{ci}(t)$ and $p_{ci}(t)$ $(i = 2, 3)$ to the equations of motion governed by the Hamiltonian H'_ω can be obtained by transforming the primed solutions $x'_{ci}(t)$, $p'_{ci}(t)$ and equation (14) to the non-primed system. The result can be expressed as

$$\begin{aligned} x_i(t) = & - \frac{d_i}{m_i \Omega_i^2} - m_j d_i B^2 + \left[A x_i(0) + B p_j(0) + \frac{d_i}{m_i \Omega_i^2} \right] cos \Omega_i t \\ & + \frac{1}{m_i \Omega_i} \left(p_i(0) + C x_j(0) - m_i d_j B \right) sin \Omega_i t \\ & - B \left(p_j(0) + C x_i(0) - m_j d_j B \right) cos \Omega_j t \\ & - m_j \Omega_j B \left[A x_j(0) + B p_i(0) + \frac{d_j}{m_j \Omega_j^2} \right] sin \Omega_j t. \end{aligned} \tag{20}$$

and

$$\begin{aligned} p_i(t) = & A B m_i d_j + \frac{C d_j}{m_j \Omega_j^2} - A m_i \Omega_i \left[A x_i(0) + B p_j(0) + \frac{d_i}{m_i \Omega_i^2} \right] sin \Omega_i t \\ & + A \left(p_i(0) + C x_j(0) - B m_i d_j \right) cos \Omega_i t \\ & - C \left[A x_j(0) + B p_i(0) + \frac{d_j}{m_j \Omega_j^2} \right] cos \Omega_j t \\ & - \frac{C}{m_j \Omega_j} \left(p_j(0) + C x_i(0) - B m_j d_i \right) sin \Omega_j t. \end{aligned} \tag{21}$$

where $i, j = 2, 3$ $i \neq j$. It can be easily realized that if we identify $\alpha_i;(i = 2, 3)$ with the following complex combination of classical initial coordinates and velocities

$$\alpha_i = A x_i(0) + B p_j(0) + \frac{A d_i}{m_i \Omega_i^2} + i \frac{b_i^2}{\hbar} \left[p_i(0) + C x_j(0) + \frac{C d_j}{m_j \Omega_j^2} \right] \tag{22}$$

where $i = 2, 3$ $(j \neq i)$, the results for expectation values given by equations (19) and classical quantities given by equations (20), (21) become identical. In this sense, the states constructed by equation (15) may serve as the exact quantum-mechanical counterpart of a classical cranked harmonic oscillator.

Another way of seeing clearly the classical behavior of the coherent states is by looking at their coordinate representation $\psi_{\{\alpha_i\}}(\{x_i\};t)$.

$$\psi_{\alpha_2,\alpha_3}(x_2,x_3) = \langle x_2 x_3|\alpha_2\alpha_3\rangle = \int_{-\infty}^{\infty} dX_2' \int_{-\infty}^{\infty} dX_3' \,\langle x_2x_3|X_2'X_3'\rangle \,\langle X_2'X_3'|\alpha_2\alpha_3\rangle\,. \quad (23)$$

The functions $\langle X_2'X_3'|\alpha_2\alpha_3\rangle$ are the known oscillator eigenfunctions of the Hamiltonian H' in the X_2' X_3' representation, which is given by:

$$\psi_{\{\alpha_i\}}(\{X_i'\};t) = \prod_{i=1}^{3} exp\left(-\frac{|\alpha_i|^2}{4b_i^2}\right)\sum_{n_i=0}^{\infty}\left(\frac{\alpha_i}{\sqrt{2}b_i}\right)^{n_i}\frac{1}{\sqrt{n_i!}}e^{-i\Omega_i\left(n_i+\frac{1}{2}\right)t}\phi_{n_i}(X_i') \quad (24)$$

where, the function $\phi_{n_i}(X_i')$ is eigenfunction of a one-dimensional harmonic oscillator.

Using the generating formula for Hermite polynomials

$$\sum_{n=0}^{\infty}\frac{s^n}{n!}H_n(x) = e^{-s^2+2sx} \quad (25)$$

the summations in equation (24) can be performed to obtain

$$\psi_{\{\alpha_i\}}(\{X_i'\};t) = \prod_{i=1}^{3}\left(\frac{|m_i|\Omega_i}{\pi\hbar}\right)^{1/4} e^{-\frac{1}{2}i\Omega_i t} \times exp\left(-\frac{1}{2b_i^2}(X_i'^2 - 2\alpha_i(t)X_i' + \alpha_i^R(t)\alpha_i(t)\right) \quad (26)$$

in the x_2x_3 representation. Analogously, because of the hermitian metric of the probability amplitudes its complex conjugate is eigenfunction to the operator x_2, x_3 given by equations (7) in the $X_2'X_3'$ representation. Exploiting these properties one obtains by solving differential equations:[2]

$$\phi_{X_2'X_3'}(x_2,x_3) = \langle x_2x_3|X_2'X_3'\rangle = \frac{1}{2\pi\hbar|B|}exp\left(\frac{i}{\hbar B}(-Ax_2x_3 + x_2X_3' + x_3X_2' - X_2'X_3')\right) \quad (27)$$

where the normalization factor is chosen such that transformation (23) is unitary. Using the equations (26), (27) and performing some gaussian integrals, normalized eigenfunction (23) gives

$$\psi_{\{\alpha_i\}}(\{x_i\};t) = Nexp\left(-\gamma_2^2x_2^2 - \gamma_3^2x_3^2 + \delta_2x_2 + \delta_3x_3 + \eta x_2x_3 + \zeta\right) \times e^{-\frac{1}{2}i(\Omega_2+\Omega_3)t} \quad (28)$$

where

$$
\begin{aligned}
N &= \sqrt{\frac{D}{\pi b_2 b_3}} exp \left(-\frac{\alpha_2^R(t)\alpha_2(t)}{2b_2^2} - \frac{\alpha_3^R(t)\alpha_3(t)}{2b_3^2} \right) \\
\gamma_i^2 &= \frac{D}{2b_i^2} \\
\delta_i &= \frac{D}{b_i^2} \left(\alpha_i(t) + i\hbar B \frac{\alpha_j(t)}{b_j^2} \right) \\
\eta &= -\frac{i}{\hbar B}(A - D) \\
\zeta &= \frac{\hbar B D}{b_2^2 b_3^2} \left[\frac{\hbar B}{2} \left(\frac{b_2^2 \alpha_3^2(t)}{b_2^2 b_3^2 + \alpha_2^2(t)} - i\alpha_2(t)\alpha_3(t) \right) \right] \\
D &= \left(1 + \frac{\hbar^2 B^2}{b_2^2 b_3^2} \right)^{-1}
\end{aligned}
\tag{29}
$$

The probability density of this coherent states can now be brought to the form

$$
|\psi_{\alpha_2\alpha_3}(x_2, x_3; t)|^2 = |N|^2 \prod_{i=2,3} exp \left[-\frac{D}{b_i^2} \left(x_i - x_{ci}(t) \right)^2 \right] \tag{30}
$$

where, the $x_{ci}(t)$ $(i = 2, 3)$ are given by equations (19), and factor $|N|^2$, where N is new normalization constant, is

$$
|N|^2 = \frac{D}{\pi b_2 b_3} \tag{31}
$$

This indicates that $|\psi_{\alpha_2\alpha_3}(x_2, x_3; t)|^2$ is a gaussian probability distribution whose center follows the classical trajectory for the cranked anisotropic oscillator in the $x_2 x_3$ plane. This supports the fact that $\psi_{\{\alpha_i\}}$ is really a coherent state for this system in the usual sense.

References

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