Classical mechanics for the circle-step potential

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1 Trajectories going through the step

We describe here the structure of the manifolds where the trajectory allways passes the step and continue on the other side. Many points in the phase space then only have a short future and past because the trajectory is pruned (forbidden) because of total reflection. The other pruning mechanisms results from trajectories which are reflected from the circle at the points $x = 0, y = \pm 1$, and from a smooth folding of the phase space which are more difficult to directly relate to a simple geometrical cause.

A picture of unstable and stable manifolds are given in figure 1. We have in this figure also drawn a number of curves correspoinding to singular trajectories. Curves R_1 and R_2 are the trajectories which when going into the right semicircle will be on the border of having a total reflection the next time it hits the $x = 0$ line. The area R between these two border curves do then not contain any part of the unstable manifold. The manifolds approaches the curves R_1 and R_2 from the outside and stop discontinous on the curve. This is analogous to the way manifolds changes in the corner of a disk billiard, or the wedge billiard [?]. For this parameter value there is only trajectories which bounce *once* in the semicircle that can get a total reflection. For higher parameter values there may be several total reflection regions.

The other curves drawn in the figure, labelled B_1, B_2, B_3, $\ldots B_n$, $\ldots$, are the trajectories which move into the right semicircle and bounces n times in the semicircle before it hits the point $(x = 0, y = -1)$. At this point the manifolds are continuous but change direction. This is analogous to the way manifolds changes at tangent orbits in dispersive billiards and at the singular point in the stadium billiard [?]. In figure 2 we show how trajectories fold around this singular point. The dotted curve bounces on the right side and the dashed curve bounces on the left side of the corner point. After the bounce only the dotted curve crosses the corner point curve and at the next bounce both curves are on the left side of the corner bouncing trajectory.

The third mechanism in the flow is a smooth folding of the phase space visable in figure 1 as a smooth S-shapes in the manifolds. In figure 3 we have drawn three

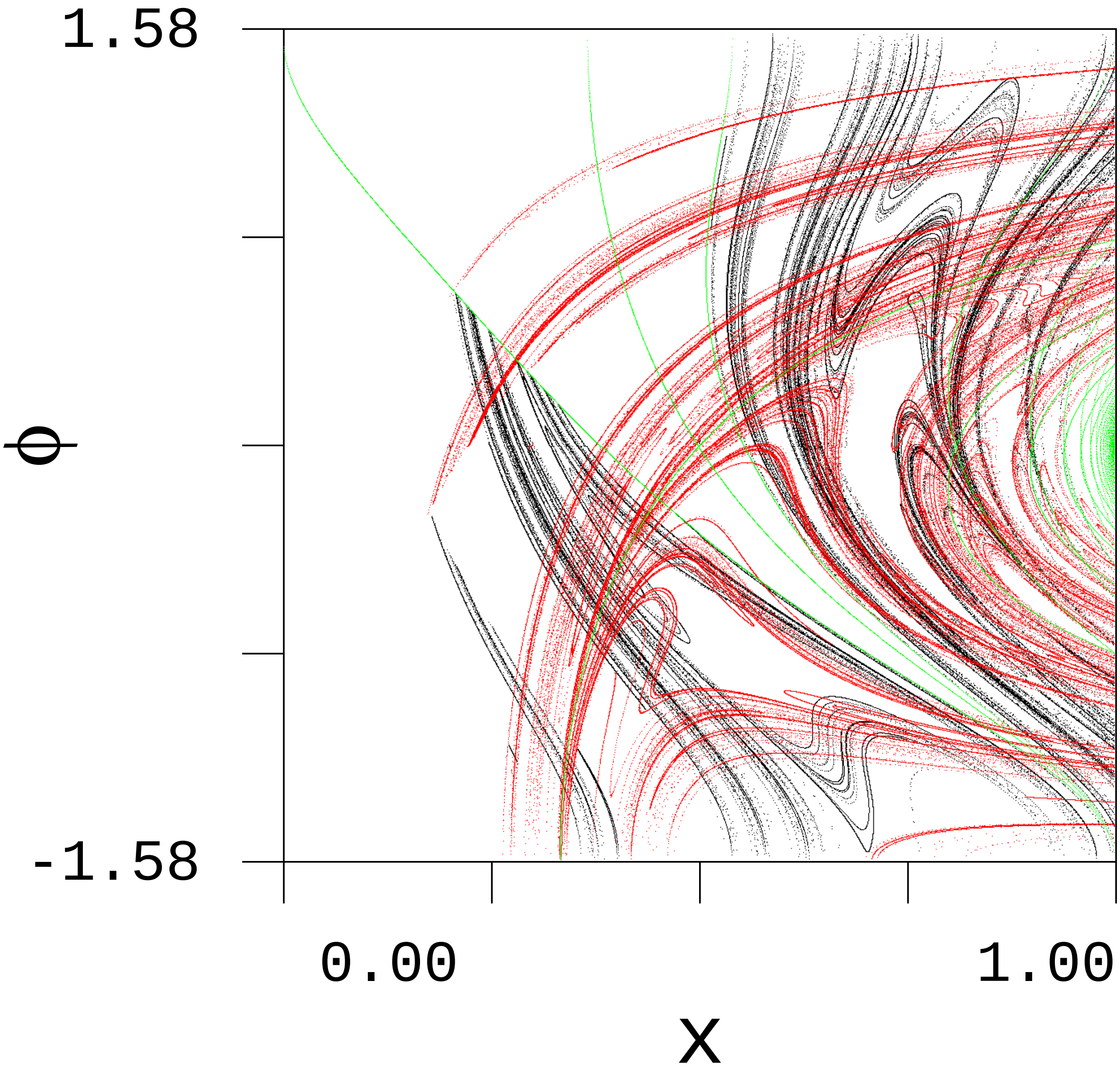

Figure 1: The stable and unstable manifolds of the map.

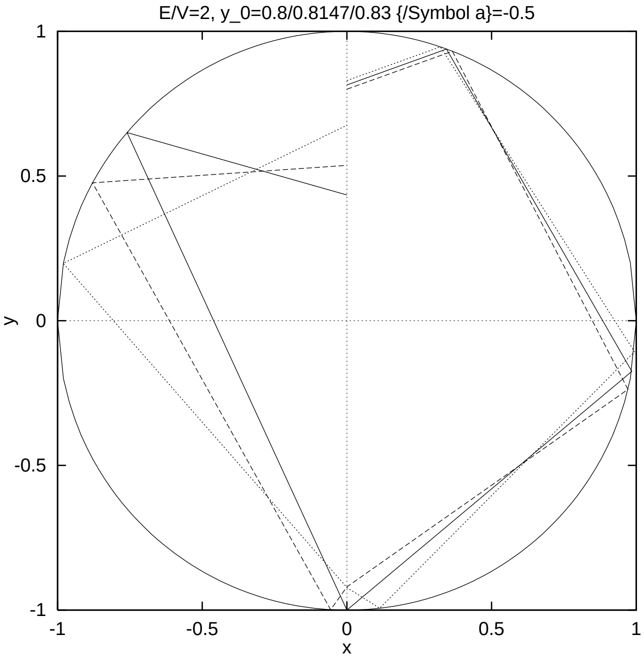

Figure 2: Trajectories bouncing on both sides and on the corner point illustrating how one gets a sharp folding of the phase space here

trajectories starting from three different points of the same fold of the unstable manifold but on three different parts of the fold separated by a folding point (a primary homoclinic tangency). Toghether with each trajectory we have drawn a neighbor trajectory starting with a slightly larger initial angle α. We have to follow the trajectory for a rather long time before we find that the orientation of the two neighbor trajectories are the same for the uppermost and downmost trajectories, while the one in the middle have a different orientation. This is a similar mechanism as we found for the corner singularity, but here we have to singular points and the at the singular points the manifolds changes smoothly, not a sharp change in the direction of the manifolds. The singular points are here homoclinic tangencies between the stable and unstable manifold. This is the mechanism which creates stable orbits, and we do find the stable orbits in this region. This structure is typical for smooth systems and billiard systems which are not completely chaotic, se e.g. [**?**, **?**, **?**, **?**].

There is a smooth S-shape folding for each region separated with the B_n curves and each has to be threated separately. In each B region we then get two partition curves which are a connected curve through the primary homoclinic tangencies. We denote these curves H_0^a, H_0^b, H_1^a, H_1^b, H_2^a, H_2^b, $\ldots$, H_n^a, H_n^b.

To construct a symbolic dynamics we now have to introduce different symbols for each region separated by curves through the singular orbits. We choose to give these a simple enumbering such that $s = 0$ is the region with $y < H_0^a$, $s = 1$ is

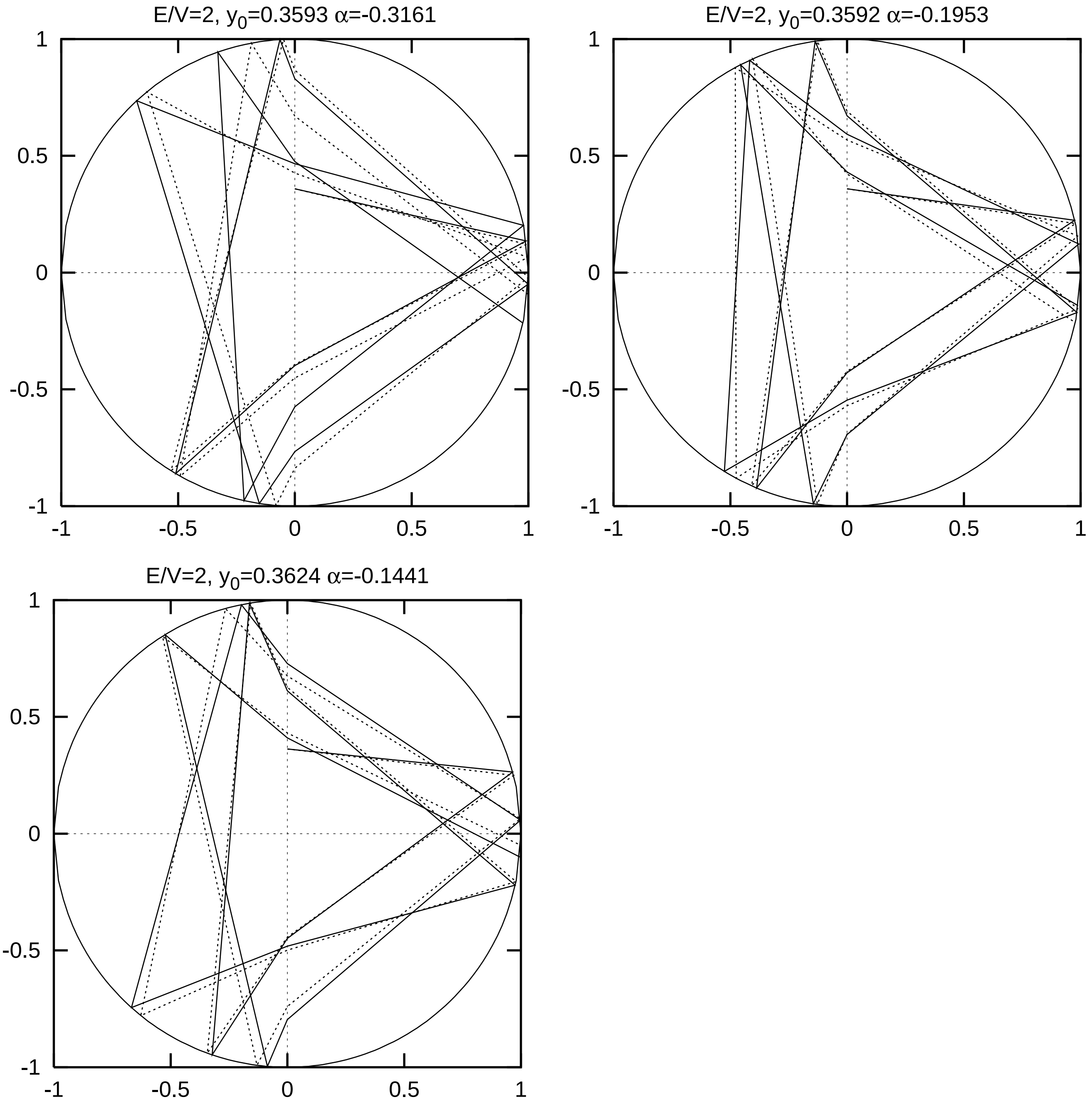

Figure 3: Trajectories with neighbor trajectories from the lower, middle and upper part of region A. The orientation between two neighbor trajectories are different in the middle part compared with the upper and lower case where the trajectories end. (a) $y_0 = 0.3593$, $\alpha = -0.3161$, (b) $y_0 = 0.3592$, $\alpha = -0.1953$, (c) $y_0 = 0.3624$, $\alpha = -0.1441$.

the region $H_0^a < y < H_0^b$, $s = 3$ if $H_0^b < y < R_1$, $s = 4$ if $R_2 < y < H_1^a$, $s = 5$ if $H_1^a < y < H_1^b$, $s = 6$ if $H_1^b < y < B_2$, $\ldots s = 3n + 1$ if $R_n < y < H_n^a$, $s = 3n + 2$ if $H_n^a < y < H_n^b$, $s = 3n + 3$ if $H_n^b < y < B_{n+1}$.