# Two-disk Pinball Game In a Weak Gravitational Field

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## Abstract

A two-disk pinball system in a weak gravitational field is studied. The addition of gravitational field suffices to makes the two-disk pinball system chaotic. The symbolic dynamics is constructed, the linear stability studied and the escape rate are calculated by the periodic orbit theory.

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## I. INTRODUCTION

Transport is a basic concept in the statistical mechanics, and there are a lot of transport phenomena in the nature [1, 4]. For example, big molecules move in the net of polymer, and the valence electrons move through a lattice formed by the positive ions.

Now, the two-disk pinball game in a weak gravitational field is a mini version of the transport problem. Though it is simple, it has the character of the transport problem. Using the periodic orbit theory, we have already calculated some transport properties of the system, such as the Lyapunov exponent, the escape rate (the average staying time of the pinball), when there is no external field affecting the pinball's moving in the two-pinball game [1, 2].

When a gravity field is added, the pinball system will no longer have the rotational symmetry and the trajectories of the pinball between disks are now parabolic segments instead of straight lines.

## II. TWO-DISK PINBALL SYSTEM

We consider a 2-D system. A straight line of length $R$ lies horizontally in the plane of the paper. We place the coordinate origin at the center of the line, so, the coordinates of the two ends are $A(R/2, 0)$ and $B(R/2, 0)$, as shown in the Fig. 1. The gravitational field points perpendicularly downward. Two disks with radii $a$, are put at the ends of the straight line.

An ideal point-size pinball of mass $m$ enters the system with velocity $v_0$. The collisions between the pinball and the disks are elastic and instantaneous.

Now, we need to decide the area in which the pinball moves in the system. The system has 3 fixed geometrical boundaries, $x = \frac{R}{2} + a$, $x = -(\frac{R}{2} + a)$, and $y = -a$. When the pinball crosses the boundaries from inside the system, it cannot return. The upper boundary is fixed by the maximal potential energy of the pinball.

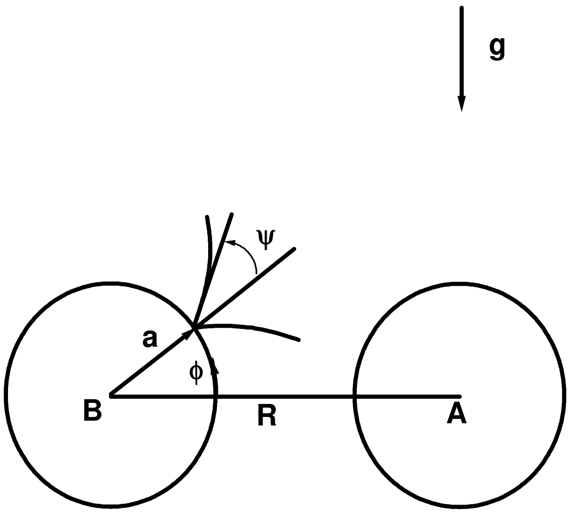


FIG. 1: The configuration of the disks and the direction of the gravity.

## III. THE GOVERNING EQUATIONS

Between the collisions, the pinball moves according to the gravitational Newton equation,

$$\begin{cases} \dfrac{d^2x}{dt^2} = 0 \\ \\ \dfrac{d^2y}{dt^2} = -g \end{cases} . \tag{1}$$

where $g$ is the gravitational accelation constant.

During the collision, the energy and the momentum of the pinball are conserved. Assume $\mathbf{n}$ is the local normal direction of the collision place. The velocities after the collision are given by,

$$\mathbf{V}_{after} = \mathbf{V}_{before} - 2(\mathbf{V}_{before} \cdot \mathbf{n})\mathbf{n}. \tag{2}$$

## IV. NONDIMENSIONALIZATION OF THE SYSTEM AND THE CHOICE OF THE PARAMETERS

Rescale (1) as $x = x'L$, $y = y'L$ and $t = t'T$:

$$\begin{cases} \dfrac{d^2x}{dt^2} = 0 \\ \\ \dfrac{d^2y}{dt^2} = -\dfrac{gT^2}{L} \end{cases} ; \tag{3}$$

The system has 5 parameters: the length of the straight line $R$, the radii of the disk $a$, the mass of the pinball $m$, the total energy of the pinball $E$, and the gravitational acceleration $g$.

Noting that

$$-\frac{gT^2}{L} = g\frac{L}{(\frac{L}{T})^2},$$

we can let $L = a$ and $T = a\sqrt{\frac{m}{2E}}$, reduce the number of the parameters of the system to 2: $r = R/a$ and $G = \frac{mga}{2E}$.

After the nondimensionalization, the radii of the disks is $a = 1$. The length of the straight line is $R = r$. The coordinates of the two ends are $A(0.5r, 0)$ and $B(-0.5r, 0)$. The equations of motion are,

$$\begin{cases} \dfrac{d^2x}{dt^2} = 0 \\ \\ \dfrac{d^2y}{dt^2} = -G \end{cases} \tag{4}$$

Suppose the initial velocity of the pinball is $v_0$. If the initial position of the pinball is $(x_0, y_0)$, by energy conservation, $v_0 = \sqrt{1 - 2Gy_0}$.

## V. TRAJECTORIES AND THE POICARÉ SECTION

The simulation of the trajectories of the pinball between collisions can be done directly by integrating the newton equation (4). After the determination of the collision points of the pinball with the disk, we can get the reflected velocity by equation (2). The typical trajectories of the pinball are shown in Figs. 2 and 3. Fig. 2 shows that when the pinball moves between the 2 disks in a weak gravity field, the trajectories are similar to that when there is no gravity field. However, when the pinball moves over the 2 disks, the trajectories are similar to that of projectiles.

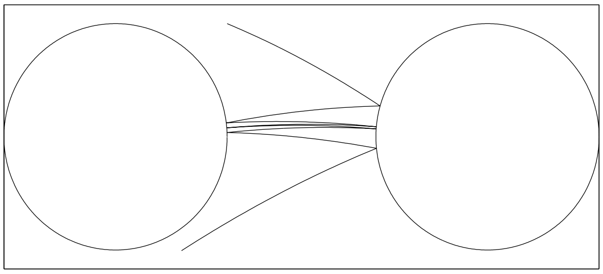

FIG. 2: One of the typical trajectories containing lines with small curvature

One of the choices of the Poincaré section is to mark the location and the components of the velocity perpendicular to the normal of the collision point, $v_{\perp} = v \sin \psi$ . By $x \rightarrow -x$ symmetry, it suffices to plot the Poincaré section of one of the disks, in our case, we plot the Poicaré section of the disk 2. The phase space of the pinball is 4 dimensional; after taking

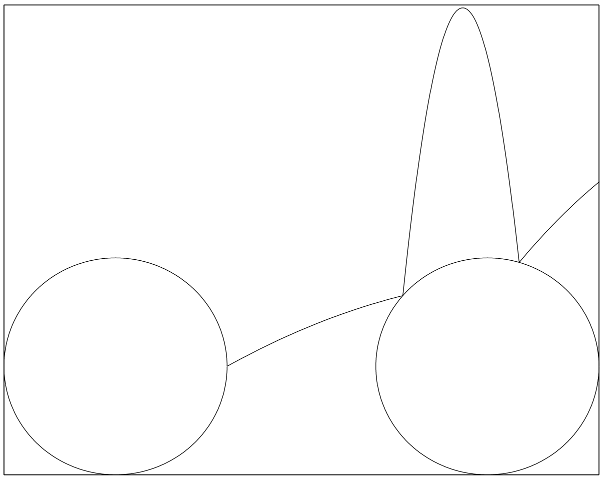

FIG. 3: New type of trajectory that exists for non-zero gravitational acceleration .

the Poincaré section, it becomes 3 dimensional; and because of the conservation of the total energy, the independent coordinates are actually only 2. We can take the polar angles of the pinball on the disks as one of the coordinates, denoted as $\phi$, and the reflected angle of the pinball with the normal as the other coordinates, denoted as $\psi$. We take the anti-clockwise direction as the positive direction. Fig. 4 is the Poincaré section for disk 2.

## VI. LINEAR STABILITY OF CYCLES

The phase space coordinates of trajectories can be specified as $(x, y, v\sin\theta, v\cos\theta)$, where $v$ is fixed by the conservation of total energy and $v = \sqrt{1-2Gy}$. Now, the $3-d$ neighborhood of the trajectories can be represented as $\delta = (\delta d_\perp, \delta d_{||}, \delta\theta)$, where $\delta d_\perp$, $\delta d_{||}$ are the variations perpendicular and parallel to the local trajectories.

Hence, the matrix of variations for trajectories between collisions will be

$$A = \begin{pmatrix} 0 & 0 & v \\ \frac{G}{v}\sin\theta & -\frac{G}{v}\cos\theta & 0 \\ 0 & 0 & \frac{G}{v}\cos\theta \end{pmatrix}, \tag{5}$$

and the Jacobian matrix for the collision is

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{2}{\cos\psi} & 0 & -1 \end{pmatrix}. \tag{6}$$

When the pinball moves between collisions, the Jacobian matrix can be calculated along the trajectories by a convenient numerical integrator.

## VII. SYMBOLIC DYNAMICS AND PERIODIC ORBITS

In a gravity field, for any two points in a two dimensional space, when the magnitude of the initial velocity is fixed, there has either no trajectory or two parabolic trajectories connecting them. One of the trajectories has the minimal flight time and the other has the maximal flight time. So, in our two-disk pinball game, according to the initial and final positions of trajectories, and the flight time, there are 4 kinds of trajectories between reflections. The first are trajectories connecting different disks with maximal flight time; the second are trajectories connecting different disks but with minimal flight time; the third are trajectories connecting the same disk with maximal flight time; and the fourth are trajectories connecting the same disk with minimal flight time. We denote them as 0,1,2 and 3 respectively. Thus, we have constructed the symbolic dynamics. The pruning rules are changed with the parameters $r$ and $G$, for example, when $G$ is small enough, the fourth kind of trajectories "3" can not exist. These symbols and the labels of the disks are combined together into strings to label the cycles. The first character of each string is the label of one of the disks 1 or 2, which is the smallest label number of the disks the cycle visits; the other characters of each string are the symbols of the trajectories. Each string represents a complete cycle following the specified itinerary with least time. Sometimes, the symbol strings represent trajectories with twice of the length of the string. For example, as shown in the Fig. 5, the cycle 1 is actually cycle 11.

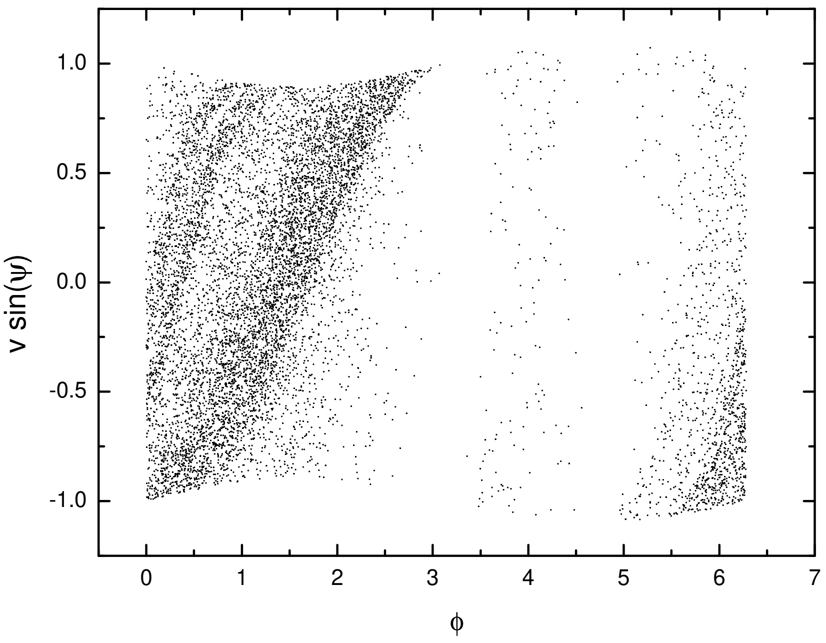


FIG. 4: The Poincaré section on the surface of the disk 2, where $r = 6$, $G = 0.1$

It is convenient to search the cycles using the least action method. The variables of the action function are the hitting positions of the pinball to the disks, denoted by $\mathbf{X}_i$, where $i = 1, 2, 3, ..., n$, when the length of the cycle is $n$. There is a trajectory connecting $\mathbf{X}_i$ and $\mathbf{X}_{i+1}$ when $i < n$, and connecting $\mathbf{X}_n$ and $\mathbf{X}_1$ provided that the flight time of the trajectories are specified. But when specified the hitting points first, the incident and the reflected angles are not necessarily equal to each other for a cycle. Since for a real cycle they are equal to each other, we can construct a cost function as,

$$L = \sum_{i=1}^{n} (\psi_i^{incident} - \psi_i^{reflected})^2. \tag{7}$$

The positions of the hitting points $X_i$'s of a cycle make $L$ reach its minimal point, in our case, which is $L = 0$. This minimization process can be done by the common 'Golden Division' method. For simplicity, we will choose $G = 0.1$ and $r = 6$. The trajectory 3 is pruned. Fig. 5 is the cycles for $G = 0.1$ and $r = 6$ up to bounce 3.

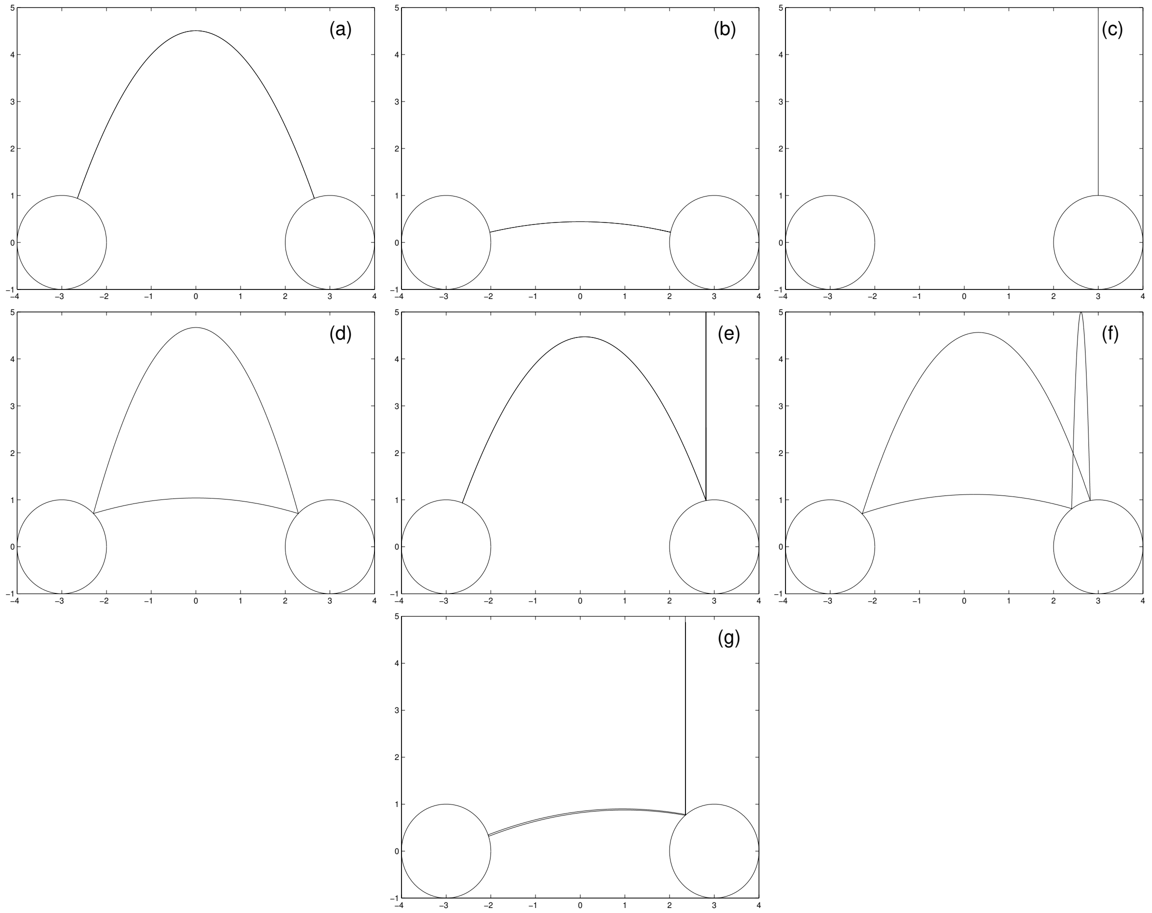


FIG. 5: The periodic orbits for $r = 6$, $G = 0.1$ up to bounce 3; (a) 10, (b) 11, (c) 12, (d) 101, (e) 1002, (f) 1012, and (g) 1112.

## VIII. DYNAMICAL ZETA FUNCTION AND THE ESCAPE RATE

Suppose $\Gamma_n$ is the survival probability of the pinball after n bounces, the escape rate of the system can be defined as the limit of the logarithm of the ratio of $\Gamma_n$ and $\Gamma_{n+1}$ as,

$$\gamma = \lim_{n \to \infty} \Gamma_n / \Gamma_{n+1}. \tag{8}$$

One the other hand,the escape rate is the logarithm of the root,which is the nearest one to 1, of the dynamic zeta function [1, 3],

$$1/\zeta(z) = \prod_p (1 - t_p),\ t_p = \frac{z^{n_p}}{|\Lambda_p|}, \tag{9}$$

where $p$ represents cycles, $n_p$ is the bounce of the cycle, the $\Lambda_p$ is the products of all the expansive eigenvalues of the Jacobian of the cycle. For our 2-disk pinball system in a weak gravitational field, the calculation of $n_p$, $\Lambda_p$ and s, the root of the zeta function truncated to specified bounced are list in Table VIII.

When we keep the cycles up to bounce 6, the root of the dynamical function of the system is $s = 9.033981526$. Hence, the escape rate is

$$\gamma = \ln 9.033981526 = 2.200993192$$

## IX. CONCLUSION

In this project, we studied the two-disk pinball system in a weak gravity field, which is a simple chaotic system. The gravitational acceleration makes trajectories of the pinball parts of parabola. When we choose the system parameters to be $r = 6$ and $G = 0.1$, because the distance between the two disks is quit large compare to the radii of the disks, and the gravity is weak enough to prune the possibility of connecting two points in a same disks with the least flight time trajectories, the symbolic dynamics turns to be simple and controllable. The escape rate is calculated by the periodic orbit theory, and we found it is quit large for the parameters we specified.

## X. ACKNOWLEDGMENTS

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[1] P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner and G. Vattay, *Chaos: Classical and Quantum* (Niels Bohr Institute, Copenhagen 2003); ChaosBook.org

TABLE I: The $n_p$, $\Lambda_p$ and s up to bounce 5 for $r = 6$ and $G = 0.1$

| $p$ | $n_p$ | $\Lambda_p$ | Q[a] | s |
|---|---|---|---|---|
| 12 | 1 | -12.93889079 | 2 | 12.94652944 |
| 10 | 2 | 1052.399834 | 1 | – |
| 11 | 2 | 103.9005819 | 1 | – |
| 101 | 2 | 509.8909321 | 2 | 10.19316331 |
| 1002 | 3 | -20319.11180 | 2 | – |
| 1012 | 3 | -19344.29859 | 4 | – |
| 1112 | 3 | -4306.028195 | 2 | 9.033985047 |
| 10001 | 4 | 531510.8553 | 2 | – |
| 10011 | 4 | 178301.5526 | 2 | – |
| 10111 | 4 | 57369.83762 | 2 | – |
| 102 | 4 | 6660.684010 | 1 | – |
| 112 | 4 | 245750809.7 | 1 | – |
| 10212 | 4 | 743143.0998 | 2 | 9.033984183 |
| 100002 | 5 | -20847619.97 | 2 | – |
| 100222 | 5 | -16700568.39 | 2 | – |
| 102022 | 5 | -1913971.762 | 2 | – |
| 100012 | 5 | -20410504.53 | 4 | – |
| 100102 | 5 | -24614256.15 | 4 | – |
| 100112 | 5 | -6943373.224 | 4 | – |
| 101102 | 5 | -20934469.92 | 2 | – |
| 101012 | 5 | -9844898.472 | 4 | – |
| 110012 | 5 | 254728123.4 | 2 | – |
| 101112 | 5 | -2198364.969 | 4 | – |
| 110112 | 5 | -4050271.204 | 4 | – |
| 111112 | 5 | -2664936.761 | 4 | – |
| 101222 | 5 | -31499698.62 | 4 | – |
| 111222 | 5 | -2799396.631 | 2 | – |
| 112022 | 5 | -26073893.95 | 4 | – |
| 112122 | 5 | -241122909.2 | 2 | 9.033984183 |



[a]$Q$ is the degeneracy of the cycles

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[4] P. Cvitanović, P. Gaspard and T. Schreiber, *CHAOS* **2**, 85 (1992).