# Two-level atom in the field of a plane standing electromagnetic wave

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December 11, 2003

**Abstract**

The motion of two level atom in the field of standing electromagnetic wave is analyzed. Poincare sections are calculated. Some periodic orbits and their stabilities found. Convergence of the cycle expansion is discussed.

## 1 Introduction

The motion of the atom in the single mode electromagnetic field is one of the basic problems in atomic physics. This problem appears, for example, in the atomic trapping and cooling [1], and in the experiments with optical cavities, when the interaction of a single atom with the field in high finesse cavity is studied [2].

In this project we are going to study a two level atom in the field of plane standing electromagnetic wave $\vec{E} = \hat{e}_z E \cos kx \cos \omega t$. The state of atom is described by the wave function

$$\Psi(x,t) = A(x,t)|1\rangle + B(x,t)|2\rangle$$

where $|1\rangle$ and $|2\rangle$ are the ground and excited states of the atom. One can always consider the two level system as the spin $1/2$ particle. Introducing the pseudo spin components

$$\begin{aligned} 2s_x = u &= P_+ \cos \omega t + P_- \sin \omega t \\ 2s_y = v &= P_+ \sin \omega t + P_- \cos \omega t \\ 2s_z = w &= |B|^2 - |A|^2, \end{aligned} \tag{1}$$

where $P_\pm = A^* B \pm AB^*$ one can write the optical Bloch equation for the evolution of this vector as follows.

$$\frac{d\vec{s}}{dt} = [\vec{N} \times \vec{s}]$$

where $\vec{N} = (\Delta', 0, \Omega \cos(kx'))$, $\Delta$ is the detuning of the laser frequency from the atomic resonance and $\Omega = |\vec{E}\vec{d}|/\hbar$ is the Rabi frequency. The Newton law gives the following equation for the spatial coordinate of the atom.

$$m\ddot{x}' = \frac{1}{2}\hbar\Omega \sin(kx') \tag{2}$$

Using new "dimensionless" variables $x = kx'$, $t' = \Omega t$ we can write the following set of equations

$$\begin{aligned}
\dot{u} &= -\Delta v \\
\dot{v} &= \Delta u + w \cos x \\
\dot{w} &= -v \cos x \\
\ddot{x} &= \epsilon^2 u \sin x.
\end{aligned} \tag{3}$$

Here $x$ is the coordinate of particle, $\Delta = (\omega_{21} - \omega)/\Omega$ is dimensionless detuning of radiation frequency from the resonance and $\epsilon = (\hbar\omega^2/2mc^2\Omega)^{1/2}$.

The system has two integrals of motion.

$$S_1 = u^2 + v^2 + w^2 \tag{4}$$

corresponds to the unitary conditions and

$$S_2 = \frac{\dot{x}^2}{2} + \epsilon^2 ucosx - \epsilon^2 \Delta w \tag{5}$$

corresponds to the conservation of energy. Therefore in fact our system has only three independent variables. The system has also three parameters $\Delta$, $\epsilon$ and energy $S_2$. According to [6] the system can exhibit the chaotic behavior if for example $\Delta \sim \epsilon$.

# 2 Discussion of the model

The Newton equation of motion given above is valid if recoil energy $\hbar^2 k^2/2m$ is much less than the depth of our optical potential $\sim \hbar\Omega$. In other words parameter $\epsilon^2 = \hbar\omega^2/2mc^2\Omega <<$ 1.

The optical Bloch equation does not take into account the spontaneous emission. Therefore if we want this approximation to be valid, the rate of spontaneous emission should be much less than the Rabi frequency $\Omega$.

## 2.1 Standing wave in the free space

For *Rb* atom ($m = 1.44 \cdot 10^{-25} g$, $\omega = 2\pi \cdot 3.85 \cdot 10^{12} Hz$), and values of parameters $\epsilon \sim 10^{-2}$, $\Delta = 10^{-2}$ the Rabi frequency is $\Omega = \frac{\hbar\omega^2}{2c^2\epsilon^2 m} \simeq 10^8 Hz$ and detuning $\Delta' = 10^6 Hz$ (this is compatible with the diode laser line-width). This Rabi frequency corresponds to the light intensity $\sim 60 mW/cm^2$. Unfortunately in this case the Rabi frequency $\Omega$ has the same order of magnitude as the rate of spontaneous emission and the requirement $\gamma << \Omega$ is not met.

## 2.2 Standing wave in the high finesse cavity

It is known [3] that the rate of spontaneous emission in the high finesse optical cavity can be decreased by a factor of $Q$, (where $Q$ is the cavity finesse) if the resonance frequency of atomic transition is detuned from the resonance frequency of the cavity. If we assume $Q \sim 5 \cdot 10^5$ (this cavity really exist in the Prof. Chapman lab) and $\epsilon \simeq 0.1$ than $\gamma = \gamma_{vac}/Q \sim 100 Hz$ and $\Omega \simeq 10^6 Hz >> \gamma$. In this case detuning is $\Delta = 10^5$, light intensity in the cavity is $I \sim 5 \cdot 10^{-6} W/sm^2$ and the kinetic energy of the atom is $E \sim \hbar\Omega/k_B \simeq 10^{-5} K$. (It is possible to achieve this energy using the laser cooling of atoms.)

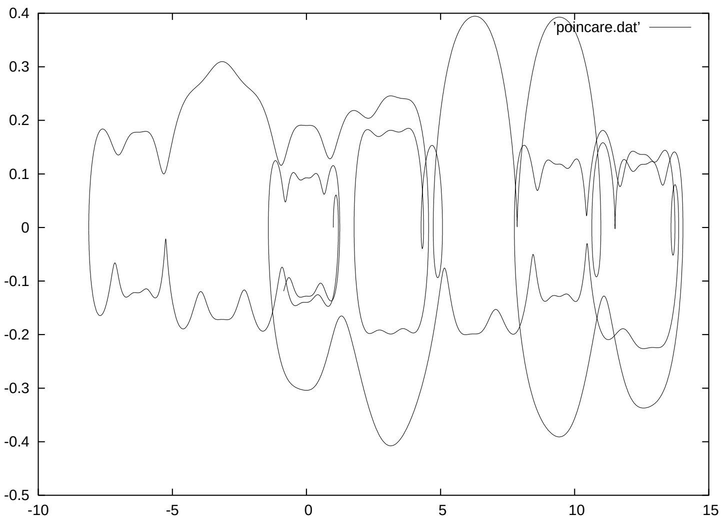


Figure 1: Trajectory of the system in $x$ (horizontal axis), $\dot{x}$ (vertical axis) coordinates for $\epsilon = 0.3$, $\Delta = 0.3$. $S_2 = 0$.

It it interesting to note that for the cavity length $10^{-4}m$ (real value) the light intensity $5 \cdot 10^{-6} W/sm^2$ corresponds to about 100 photons/$sm^2$ inside the cavity (or about 1 photon per cavity if the cavity size is $\sim 1mm^2$).

Although our model seems to work well in this case, the experimental observation of the chaotic dynamic can be a very challenging task, (see eq. [4, 5]).

# 3 Trajectories

The system of differential equations (3) was integrated using the GSL library. Integrals of motion $S_1$ and $S_2$ were used to check the precision of our integrator. Typical trajectory for $\epsilon = 0.3$, $\Delta = 0.3$. $S_2 = 0$ is shown on Fig 1.

# 4 Poincare section

Our system is periodic in $x$. And in order to find the diffusion constants it is necessary to consider solution of the system in the fundamental domain $x \subset [-\pi, \pi]$. To keep trajectory in this interval we add or subtract $2\pi$ each time the coordinate of atom crosses the $x = \pi$ or $x = -\pi$ plane.

The equations of motion of for the atomic pseudo-spin are basically the optical Bloch equations. If the position of atom $x$ is fixed then solution of this equation is the precession of unit vector $u, v, w$ about the axis $(cos(x), 0, \Delta)$. Therefore trajectory of the pseudo-vector

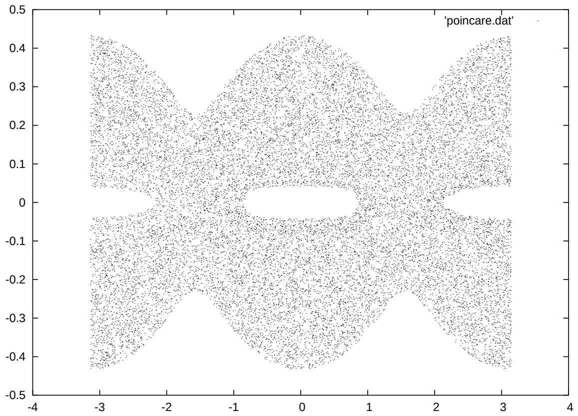


Figure 2: Poincare section $v = 0, \dot{z} > 0$ for $\epsilon = 0.3$, $\Delta = 0.3$, $S_2 = 0$.

will always cross $v = 0$ plane.

This observation is true for the moving atom as well. Precession axis will change the direction as atom moves, but this axis will always lay in the $uw$ plane. Therefore any periodic orbit should always cross $v = 0$ plane.

The Poincare sections for the different initial conditions are shown in Fig. 3 and 4.

# 5 Periodic orbits

To find periodic orbits we randomly choose the initial conditions for $x$ and $\dot{x}$ for the system on the Poincare plane with given energy. Then the system was solved for $N$-th intersection with the Poincare section. If the final point was close enough to the initial ($(\Delta x)^2 + (\Delta \dot{x})^2 < 3 \cdot 10^{-3}$), then the point was used as the initial guess and exact location of the periodic orbit was computed using the Newton method.

The combination of parameters $\Delta = 1$, $\epsilon = 1$, $S_2 = 0.6$ gives 12 periodic orbits of length 1 (see Figure 5). And there is about 800 orbits of length 4 that I have found already. Unfortunately due to lack of symbolic dynamic I can not count how many orbits I missed.

The points where the known periodic orbits crosses the Poincare plane for the periodic orbits of length 1, 2, 3, and 4 are shown on the Figure 4.

For the more realistic set of parameters $\epsilon = 0.3$, $\Delta = 0.3$, $S_2 = 0$ I was able to find 5 periodic orbits of length 1. (See Figure 6.)

Stabilities of the orbits were calculated as well. Equation for the Jacobian

$$dJ(t)/dt = AJ(t),$$

where matrix of variations $A_{ij} = \frac{\partial f_i(x)}{\partial x_j}$ and equation of motion (3) were integrated together along the periodic orbit. Then the eigenvalues of Jacobian matrix were computed. To double check our results this procedure was performed both for the original 5-dimensional system

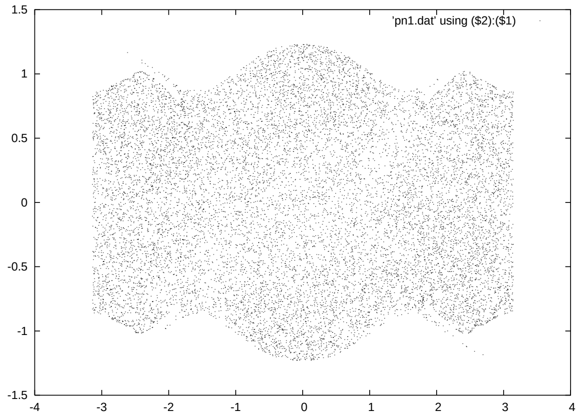


Figure 3: Poincare section for $\epsilon = 1$, $\Delta = 1$, $S_2 = 0.6$. Atom position and velocity corresponds to $x$ and $y$ axis respectively. Points corresponds to the moments when atom crosses $v = 0$ plane.

(3) and for the 4 - dimensional system, where $w$ was expressed in terms of other varibles and parameters using the energy conservation (5). In both cases we obtain the same values of two nonzero eigenvalues of the Jacobian matrix. The number of unit eigenvalues is 3 in 5 - dimensional case and 2 in 4 dimensional case. The determinant of the Jacobian always equal to one $det(J) = 1$. Therefore we have only one independent eigenvalue, and another one is given by the formula $\Lambda_2 = 1/\Lambda_1$.

List of known periodic orbits, their periods and stabilities for two different sets of parameters is given in Table 1 and Table 2. Due to the symmetry properties we usually have several different orbits that correspond to the same period $T$ and stability $\Lambda$ (orbit properties remains the same after each of these transformations: $x \leftarrow -x$, $\dot{x} \rightarrow -\dot{x}$, and $(x \rightarrow x + \pi,\ w \rightarrow -w)$. In fact it would be better to use this symmetries and consider trajectories in the fundamental domain.

It is necessary to note that in both cases there is at least 1 marginally stable periodic orbit. (The eigenvalues of the Jacobian in this case are Complex conjugate with $|\Lambda_{1,2}| = 1$).

# 6 Ideas about symbolic dynamic

Unfortunately I am not able to figure out any symbolic dynamic and count the periodic orbits. Some ideas about the symbolic dynamic are described below.

First of all variables $u$ and $v$ always oscillate with the same frequency. There is only one point where $u = 0$ between two neighboring points where $v = 0$. Variables $w$ and $\dot{x} = s$ have the different oscillation period.

One can notice that points where $v = 0$ correspond to the minimum or maximum of $w$, because $\dot{w} = -v \cos x$ is equal to zero in this point. The sign of $\ddot{w}|_{v=0} = (\Delta u + w \cos x) \cos x$ determine whether it is minimum or maximum.

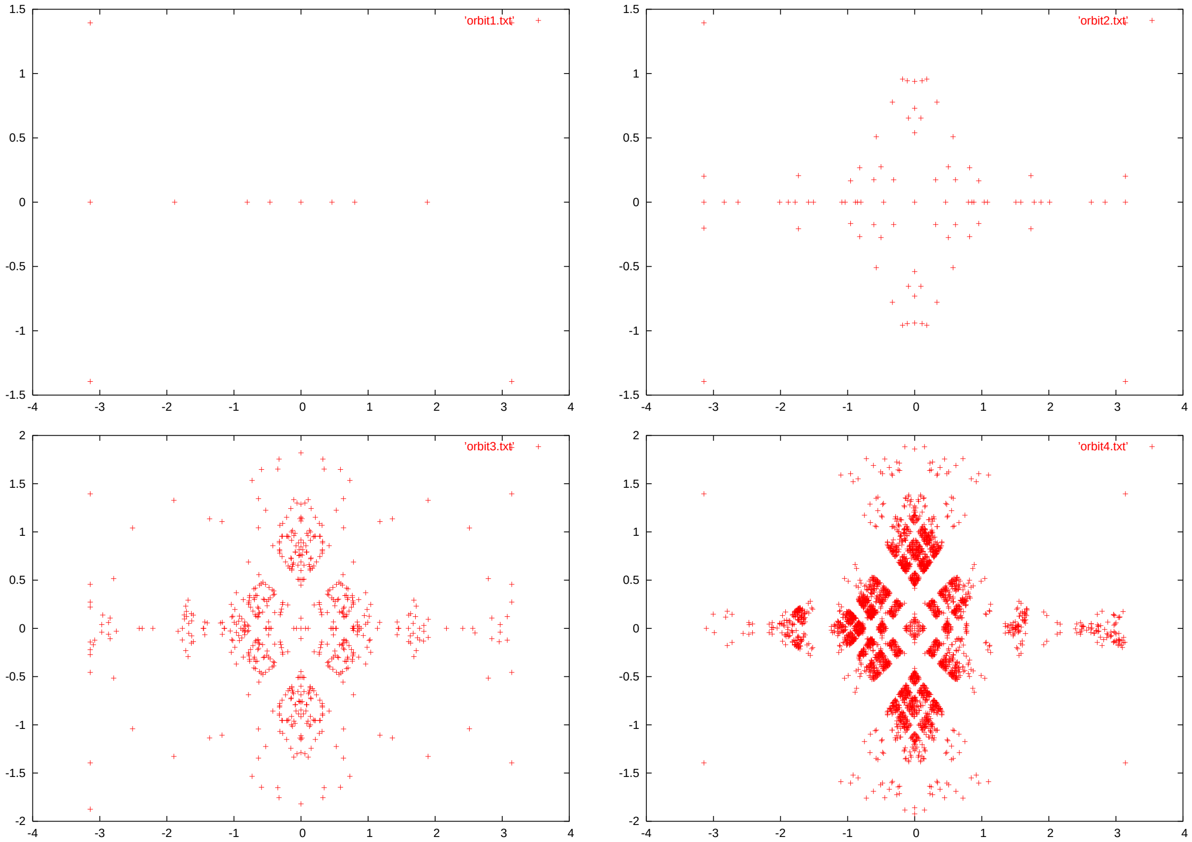


Figure 4: Points where known periodic orbits of length 1, 2, 3, and 4 crosses the Poincare section, $\epsilon = 1$, $\Delta = 1$, energy $S_2 = 0.6$. Atom position and velocity correspond to $x$ and $y$ axis respectively.

Table 1: The list of the shortest periodic orbits for $\epsilon = 0.3$, $\Delta = 0.3$, $S_2 = 0$.

| Length | Number of Orbits | Period | Stability | $n_p/2\pi$ |
|---|---|---|---|---|
| 1 | 2 | 6.018200042132e+00 | 1.000000000000e+00 | 0 |
| 1 | 2 | 1.480854492245e+01 | 1.147297510738e+02 | 0 |
| 2 | 2 | 2.812856189064e+01 | 4.175637750337e+00 | ±1 |
| 3 | 4 | 3.105884702993e+01 | 1.966948055119e+01 | ±1 |
| 3 | 4 | 3.230212474117e+01 | 8.399254977807e+01 | ±1 |
| 3 | 4 | 3.310730792960e+01 | 1.243205906812e+01 | ±1 |
| 3 | 4 | 3.348453474784e+01 | 1.000000000006e+00 | ±1 |
| 3 | 4 | 3.420434686085e+01 | 1.000000943084e+00 | ±1 |
| 3 | 4 | 3.496303343110e+01 | 3.211333408441e+02 | ±1 |
| 3 | 2 | 3.574892027792e+01 | 1.000004585045e+00 | ±1 |
| 4 | 4 | 2.949436203635e+01 | 1.931441422086e+01 | ±1 |
| 4 | 2 | 3.050479977467e+01 | 8.171855584384e+01 | ±1 |
| 4 | 4 | 3.050520723606e+01 | 7.544991654577e+01 | ±1 |
| 4 | 1 | 3.334776274932e+01 | 2.689495025512e+02 | 0 |
| 4 | 4 | 3.353438676526e+01 | 2.592629893222e+02 | ±1 |
| 4 | 4 | 3.380339726434e+01 | 2.064217816610e+02 | ±1 |
| 4 | 4 | 3.426835897857e+01 | 1.079605164532e+02 | ±1 |
| 4 | 4 | 3.442428727765e+01 | 6.699744409401e+01 | ±1 |
| 4 | 2 | 3.449903416304e+01 | 2.097595843350e+02 | 0 |
| 4 | 2 | 3.535352911464e+01 | 9.254912105720e+02 | 0 |
| 4 | 2 | 3.558948021447e+01 | 7.523726055691e+02 | 0 |
| 4 | 2 | 3.596130011382e+01 | 5.092618265125e+02 | 0 |
| 4 | 2 | 3.633957844924e+01 | 6.863748785998e+02 | 0 |
| 4 | 4 | 4.605069984334e+01 | 2.263447647195e+03 | ±1 |
| 4 | 4 | 4.612225497948e+01 | 2.239638171069e+03 | ±1 |
| 4 | 4 | 4.698794527018e+01 | 9.048327070518e+03 | ±1 |
| 4 | 4 | 4.750177031371e+01 | 7.204354515100e+03 | ±1 |
| 4 | 4 | 4.794108164753e+01 | 5.634519037465e+02 | ±1 |
| 4 | 4 | 4.804175619106e+01 | 1.003113892801e+03 | ±1 |
| 4 | 4 | 4.837562706079e+01 | 5.100231609840e+01 | ±1 |
| 4 | 4 | 4.975013448300e+01 | 3.744782863328e+04 | ±1 |
| 4 | 4 | 4.976080965643e+01 | 2.732426418902e+04 | ±1 |
| ... | | | | |

Table 2: The list of the shortest periodic orbits for $\epsilon = 1$, $\Delta = 1$, $S_2 = 0.6$.

| Length | Number of Orbits | Period | Stability | $n_p/2\pi$ |
|---|---|---|---|---|
| 1 | 2 | 4.442882938158e+00 | 8.450005865665e+00 | 0 |
| 1 | 4 | 5.242655877480e+00 | 1.000000000000e+00 | ±1 |
| 1 | 4 | 5.508651897153e+00 | 1.717985409507e+01 | 0 |
| 1 | 2 | 6.291974830365e+00 | 6.044039490689e+01 | 0 |
| 2 | 8 | 9.787250593905e+00 | 1.967772572855e+02 | ±1 |
| 2 | 4 | 1.010419318896e+01 | 1.602554199557e+02 | 0 |
| 2 | 4 | 1.079372479225e+01 | 3.634374695201e+02 | 0 |
| 2 | 8 | 1.091615145360e+01 | 3.164794956639e+02 | ±1 |
| 2 | 4 | 1.123617369958e+01 | 4.369656240490e+02 | ±1 |
| 2 | 2 | 1.128566580350e+01 | 2.258841436391e+02 | 0 |
| 2 | 4 | 1.151040939284e+01 | 2.708269055829e+02 | 0 |
| 2 | 4 | 1.156926931466e+01 | 6.703350152463e+02 | 0 |
| 2 | 2 | 1.160374099689e+01 | 2.783136822388e+02 | 0 |
| ... | | | | |

Similar arguments can be applied to $\dot{x} = s$. $s$ has minimum or maximum when $u = 0$ because $\dot{s} = \epsilon^2 u \sin x = 0$ in this point. The sign of $\ddot{w}|_{u=0} = -\Delta\epsilon^2 v \sin x$ determines whether it is minimum or maximum.

Thus one can introduce symbols $w_+$, $w_-$, $s_+$ and $s_-$ corresponding to minimum or maximum of $w$ and $s$ when trajectory crosses the planes $u = 0$ and $v = 0$ respectively. Unfortunately not all possibles sequence of symbols appears in the periodic orbit. And I can not find any pruning rules. Moreover some orbits that obviously looks different have the same symbol sequence.

# 7 Calculation of diffusion coefficient

Once the periodic orbits are found, it is possible to evaluate the diffusion coefficient using formula [8]

$$D = \frac{1}{2}\frac{\langle \hat{x}^2 \rangle_\zeta}{\langle T \rangle_\zeta} = \frac{1}{2}\frac{1}{\langle T \rangle_\zeta}\sum_p \frac{(-1)^{k+1}(\hat{n}_{p1} + ... + \hat{n}_{pk})^2}{\Lambda_{p1}...\Lambda_{pk}} \tag{6}$$

where $\Lambda_i$ is the orbit stability, $\hat{n}_i$ is the traveled distance of the atom on $i$-th periodic orbit,

$$\langle T \rangle_\zeta = \sum_p \frac{(-1)^{k+1}(T_{p1} + ... + T_{pk})}{\Lambda_{p1}...\Lambda_{pk}}, \tag{7}$$

$T$ is the periodic orbit period. In both formulas summation is over all distinct periodic orbits.

The results of this calculations using all periodic orbits up to the length 4 are given in the Tables 3 and 4. (Orbits that have stability $\Lambda = 1$ were excluded from summation.) Unfortunately for the number of terms that I use, sums here do not show any indication of convergence. Moreover one can notice that the the stabilities of the periodic orbits decrease mush slower than $\Lambda^p$, where $p$ is the orbit period. For example for $\epsilon = 0.3$, $\Delta = 0.3$ the

Table 3: The results of calculation of the diffusion coefficient for $\epsilon = 1$, $\Delta = 1$, $S_2 = 0.6$.

| Max. length | $\langle \hat{x}^2 \rangle_\zeta$ | $\langle \hat{T} \rangle_\zeta$ | $D$ |
|---|---|---|---|
| 1 | 0.0 | 2.2217 | 0.0 |
| 2 | 2.96442 | 2.1849 | 0.6784 |
| 3 | 445.815 | 25.205 | 8.8438 |

Table 4: The results of calculation of the diffusion coefficient for $\epsilon = 0.3$, $\Delta = 0.3$, $S_2 = 0$.

| Max. length | $\langle \hat{x}^2 \rangle_\zeta$ | $\langle \hat{T} \rangle_\zeta$ | $D$ |
|---|---|---|---|
| 1 | 0.0 | 0.25814 | 0.0 |
| 2 | 18.9089 | 13.7263 | 0.68878 |
| 3 | 41.3519 | 31.9515 | 0.64710 |
| 4 | 60.7077 | 42.9515 | 0.70670 |

most stable orbit of length 1 has $\Lambda = 114$, but the most stable periodic orbit of length 3 has $\Lambda = 19.3$.

# 8 Conclusion

Several reasons can influence the convergence of the cycle expansion. First of all it is possible that I missed some short periodic orbits. I can not verify it because I don't have any way to compute the total number of periodic orbits. But taking into account the number of points that I tested searching for the periodic orbits (typically about $10^6 - 10^7$ for every orbit length) I really doubt that I have missed the most stable orbits that give dominant contribution to the sum (6).

The second reason is the presence of the marginally stable trajectories. There is one marginally stable orbit of length 1 for $\Delta = 1$, $\epsilon = 1$, and there are several marginally stable orbits of length 3 for $\Delta = 0.3$, $\epsilon = 0.3$). Therefore the system can manifest intermittency, and bad convergence of the sum (6) is possible.

I would probably obtain better results if I use another approach to the periodic orbits search. Instead of finding all orbits up to the given length, I could for example search for the most stable periodic orbits regardless of its length. (Integrating the very long trajectory and identifying parts of the trajectory close to periodic orbit), and then use the stability cutoff in calculations of observables.

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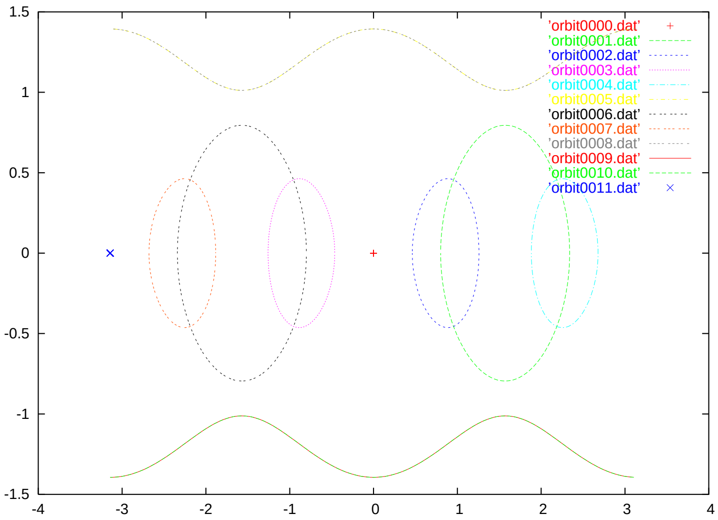


Figure 5: Some of the periodic orbits of length 1 for $\epsilon = 1$, $\Delta = 1$, energy $S_2 = 0.6$. Atom position and velocity correspond to $x$ and $y$ axis respectively. There are at least 12 periodic orbits of length 1 for this choice of parameters. The running orbit shown on the figure has the stability $\Lambda = 1$. All other orbits have the stability greater than 1.

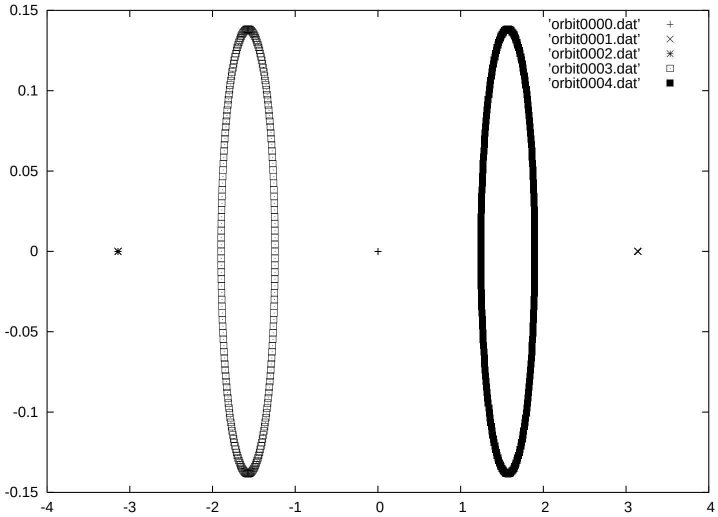


Figure 6: Periodic orbits of length 1 for $\epsilon = 0.3$, $\Delta = 0.3$, energy $S_2 = 0$. Atom position and velocity correspond to $x$ and $y$ axis respectively. There are 4 orbits of length 1 for this choice of parameters.

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