Deterministic diffusion — the sawtooth map

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This is the term paper which makes up the final part of the exam for the Fall '98 course on classical and novel approaches to non-equilibrium statistical mechanics and kinetic theory. The sole reference for the paper is the chaos web-book *Classical and Quantum Chaos: A Cyclist Treatise* by *Cvitanović & friends* at `http://www.nbi.dk/ChaosBook/`. The project description is found in appendix H.2 of the web-book.

1 Reproduction

One achievement of periodic orbit theory is the possibility to do non-equilibrium statistical mechanics in a new way, without the need for approximations, *stoßzahlansatz* or the like. It enables us in principle to relate the short term behaviour of a dynamical system to behaviour in the $t \to \infty$ limit. The cookbook recipe is simple; encode the dynamics in a suitable alphabet, calculate the great oracle: the *dynamical zeta function*, query the oracle in the right way and you will get the answers. However, the real world is a cruel place for theoretical physicists. Finding a suitable alphabet for the dynamics of real world systems is rarely easy and as with all great oracles you must approach them with respect and know exactly how to pose the questions.

Realising that the main obstacle for a young aspiring physicist is finding a sufficiently well-behaved real world problem to which he can apply newly acquired textbook knowledge, let us turn to a toy problem that can be useful for gaining insight. Let us look at something one-dimensional, linear, evolving in discrete time — in short, a problem tractible to pen and pencil calculations!

What we want to look at is the following mapping of $I = [0, 1]$ into R,

$$\hat{f}_I(\hat{x}) \stackrel{\text{def.}}{=} \begin{cases} \Lambda\hat{x}, & \hat{x} \in [0, 1/2[\\ \Lambda(\hat{x} - 1) + 1, & \hat{x} \in]1/2, 1]. \end{cases} \tag{1}$$

The function $\hat{f}_I$ is constructed such that $0 \mapsto 0$ and $1 \mapsto 1$. It is the diffusion properties of this map (and related ones to be defined below) we will investigate as function of, or rather for various values of the parameter Λ. In the following we will take this parameter to be greater than 2. We do not have the right mapping to study yet because $\hat{f}_I$ is only defined on the unit interval whereas it takes values in R. The cure is very natural and simple; extend $\hat{f}_I$ to all of R by translating the function back to the unit interval, evaluating it there, and sending it back to where it came from. In a less verbose and more mathematical formulation this translates into

$$\hat{f}(\hat{x}) \stackrel{\text{def.}}{=} \hat{f}_I(\hat{x} - \lfloor\hat{x}\rfloor) + \lfloor\hat{x}\rfloor, \tag{2}$$

where $\lfloor \cdot \rfloor$ denotes the 'floor' function which returns the nearest integer smaller than or equal to its argument. Obviously, $\hat{x} - \lfloor \hat{x} \rfloor \in I$ holds and 1 is still mapped to 1 with the extended definition (2).

Until now we have been decorating our definitions with plenty of little 'hats' just to make the reader curious as to whether any 'bald' functions would show up. Indeed they will! The convention we will follow here is to denote entities relating to the whole space (i.e. R) with hats and let 'bald' symbols refer to the elementary cell *in casu* the unit interval I. This notation is broadly that of *Cvitanović & friends*, section 14.1. If the notation is useful for no other purpose it at least makes the formulas look cool, so we will use it here.

In our aspiring physicist's recipe for doing periodic orbit theory, we mentioned something about understanding and encoding the short term dynamics in an alphabet and then using this to understand the $t \to \infty$ behaviour. This is exactly what we are going to try now. We define a new function $f : I \to I$ by translating $\hat{f}$ back to I by doing the calculations modulo unity. Likewise we get rid of the integer part of $\hat{f}(\hat{x})$. All we then have to do is to iterate f on I and keep track of the 'jumping' i.e. the discarted integer part of $\hat{f}(x)$. More precisely we define f as

$$f(x) \stackrel{\text{def.}}{=} \hat{f}(\hat{x}) - \lfloor \hat{f}(\hat{x}) \rfloor, \tag{3}$$

where $x = \hat{x} - \lfloor \hat{x} \rfloor$ is in I. We are now going to write down a symbolic dynamics where the alphabet simply keeps track of the distance of the 'jump'.

Since this is periodic orbit theory, it seems reasonable that we introduce some notion of periodicity. The notation is that of *A Cyclist Treatise* where $p = \{x_1, \ldots, x_{n_p}\}$ is called an *elementary cell cycle* (*elementary cell* = no hats) if n_p iterations sends x_j back to itself, $f^{n_p}(x_j) = x_j$. We are also to keep track of the length of the jumps so we will briefly return to the whole space, R. We define (with hats now) $\hat{n}_p \in \mathsf{Z}$ by $\hat{n}_p = \hat{f}^{n_p}(x_j) - x_j$ as the *jumping number* of the cycle. If $\hat{n}_p = 0$ the cycle is said to be *standing*, otherwise it is said to be *running*.

Now things are in place! The cycle weight for a cycle p is given in the project description (H.2) as

$$t_p(\beta, z) = z^{n_p} \frac{e^{\beta \hat{n}_p}}{|\Lambda_p|}. \tag{4}$$

Our oracle will be the dynamical zeta function given by

$$1/\zeta(\beta, z) = \prod_p (1 - t_p(\beta, z)). \tag{5}$$

The quantity we want to compute is the *diffusion constant* D which periodic orbit theory tells us is given by

$$D = \frac{1}{2} \frac{\langle \hat{n}_p^2 \rangle_\zeta}{\langle n \rangle_\zeta}. \tag{6}$$

The triangular brackets, $\langle \cdot \rangle_\zeta$, denotes mean cycle quantities, so for example $\langle \hat{n}_p^2 \rangle_\zeta$ is the *cycle mean* of the *square of the jumping number*. We are dealing with a problem in discrete time so the rôle of time is taken over by the *mean*

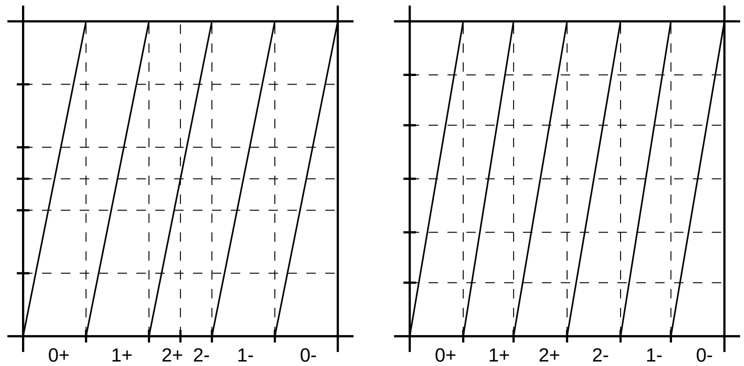

Figure 1: Illustration of the mapping f for $\Lambda = 5$ (left) and $\Lambda = 6$ (right). The jumping numbers for the intervals are indicated.

cycle length $\langle n_p \rangle_\zeta$. The mean square jumping number and the mean cycle length are given by (querying the oracle)

$$\langle \hat{n}_p^2 \rangle_\zeta = \left. \frac{\partial^2}{\partial \beta^2} \frac{1}{\zeta(\beta, z)} \right|_{\beta=0,\ z=1} \quad \text{and} \tag{7}$$

$$\langle n_p \rangle_\zeta = \left. z \frac{\partial}{\partial z} \frac{1}{\zeta(\beta, z)} \right|_{\beta=0,\ z=1} . \tag{8}$$

Let's get some real world numbers in! Out there natural numbers are very popular and it will turn out that the dynamics will be particularly simple if we choose Λ to take values in the natural numbers. We will have to consider two cases: Λ odd and Λ even. Let us start out by considering the case where Λ is odd.

Sawteeth for Λ *odd*

We will now have to understand the short term dynamics of the system, choose our alphabet, and then we can calculate the big D. The dynamics is illustrated on figure 1 for the case of $\Lambda = 5$. The interval I is partitioned into six subintervals $\{\mathcal{M}_{0+}, \mathcal{M}_{1+}, \mathcal{M}_{2+}, \mathcal{M}_{2-}, \mathcal{M}_{1-}, \mathcal{M}_{0-}\}$ corresponding to the six different possible jumping numbers. In general for Λ odd we have $\{\mathcal{M}_i \mid i = m-, \ldots, m+\}$ where m is given by $(\Lambda - 1)/2$.

We now need to choose our alphabet $\mathcal{A}$. From the figure we see that $\mathcal{M}_{0+}, \mathcal{M}_{1+}, \mathcal{M}_{1-}$ and $\mathcal{M}_{0-}$ are mapped onto the entire unit interval I. This is easily generalised to arbitrary odd Λ, where $\mathcal{M}_{0\pm}, \ldots, \mathcal{M}_{(m-1)\pm}$ are mapped onto the entire interval. The subintervals $\mathcal{M}_{2+}$ and $\mathcal{M}_{2-}$ ($\mathcal{M}_{m+}$ and $\mathcal{M}_{m-}$ in the general case of Λ odd) are only mapped onto $\mathcal{M}_{0+} \cup \mathcal{M}_{1+} \cup \mathcal{M}_{2+}$ and $\mathcal{M}_{0-} \cup \mathcal{M}_{1-} \cup \mathcal{M}_{2-}$ respectively. In the general case we get that $\mathcal{M}_{m+}$ is mapped onto $\bigcup_i \mathcal{M}_{i+}$ and similarly for $m-$.

We can write the general case out in the infinite alphabet

$$\mathcal{A} = \{(m+)^k 0+, (m+)^k 1+, (m-)^k 0-, (m-)^k 1- \mid k = 0, 1, 2, \ldots\} \tag{9}$$

in which the dynamics is unrestricted, i.e. all combinations of *letters* are possible itineraries for points. Two possibilities are not accounted for, however. The dynamics has not taken into account that it is possible for a point to be mapped from $\mathcal{M}_{m+}$ to $\mathcal{M}_{m+}$ (and similarly for $\mathcal{M}_{m-}$) *ad infinitum*. This can be taken care of rather simply by introducing factors of $(1-t_{m+})$ and $(1-t_{m-})$ in the dynamical zetafunction as we will see below.

The dynamical zeta function, $1/\zeta$, can now be calculated,

$$\begin{aligned} 1/\zeta &= \prod_p (1-t_p) \qquad (10)\\ &= (1-t_{m+})(1-t_{m-})(1-\sum_{a=0}^{m-1}\sum_{k=0}^{\infty}(t_{m+})^k t_{a+} - \sum_{a=0}^{m-1}\sum_{k=0}^{\infty}(t_{m-})^k t_{a-}). \end{aligned}$$

We have here used relations of the type $t_{(m+)^k a+} = (t_{m+})^k t_{a+}$ which makes the 'curvature corrections' of eq. (9.5) in *A Cyclist Treatise* vanish. The inner summations are easily taken care of with the aid of,

$$\sum_{k=0}^{\infty}(t_{m+})^k t_{a+} = \frac{t_{a+}}{1-t_{m+}} \qquad (11)$$

and similar expressions for t_{m-} and t_{a-}.

We can now plug into (10)

$$\begin{aligned} 1/\zeta &= (1-t_{m+})(1-t_{m-}) - (1-t_{m-})\sum_{a=0}^{m-1} t_{a+} - (1-t_{m+})\sum_{a=0}^{m-1} t_{a-}\\ &= 1 - t_{m+} - t_{m-} - \sum_{a=0}^{m-1} t_{a+} - \sum_{a=0}^{m-1} t_{a-} + t_{m-}\sum_{a=0}^{m-1} t_{a+} + t_{m+}\sum_{a=0}^{m-1} t_{a-} + t_{m+}t_{m-} \end{aligned}$$

and obtain the oracle we want to query. We do this by applying the operations $z\frac{\partial}{\partial z}$ and $\frac{\partial^2}{\partial \beta^2}$ to $1/\zeta(\beta, z)$ and evaluating at $(\beta, z) = (0, 1)$. The calculations are simplified if we note that t_p is an *eigenfunction* of the two operators with *eigenvalues* n_p and $\hat{n}_p^2$ respectively. Furthermore, we will need that $t_p(0,1) = 1/\Lambda_p$.

Straightforward (albeit tedious) calculations and the formulas (7) and (8) give

$$\begin{aligned} \langle \hat{n}^2 \rangle_\zeta &= \frac{(\Lambda-1)(\Lambda+1)(1-\Lambda)}{12\Lambda} \qquad (12)\\ \langle n \rangle_\zeta &= \frac{(1-\Lambda)}{\Lambda}, \qquad (13) \end{aligned}$$

where we have used the defining relation $m = (\Lambda - 1)/2$ to simplify the result.

The diffusion constant D now follows readily,

$$D = \frac{1}{2}\frac{\langle \hat{n}^2 \rangle_\zeta}{\langle n \rangle_\zeta} = \frac{(\Lambda+1)(\Lambda-1)}{24}. \qquad (14)$$

Voila!

Sawteeth for Λ even

Likewise and a bit simpler, we can deal with the case of even values of Λ. The interval can be partitioned into Λ equal intervals $\{\mathcal{M}\}_{i=m-}^{m+}$ where $m = \Lambda/2 - 1$. We encode the symbolic dynamics in the alphabet $\mathcal{A} = \{0+, \ldots, m+, m-, \ldots, 0-\}$. Since f is an onto mapping of the $\mathcal{M}_i$ onto the unit interval, $f(\mathcal{M}_i) = I$, the symbolic dynamics is unrestricted in the alphabet $\mathcal{A}$.

The zeta function now follows from the definition as

$$1/\zeta = 1 - \sum_{i=m-}^{m+} t_i, \qquad (15)$$

where we have used the fact that all higher terms in the sum vanish exactly: in the lingo of periodic orbit theory all curvature correction vanish because the shadowing is exact.

We now just have to crank the wheel and grind out the big D. The mean cycle length is calculated as

$$\langle n \rangle_\zeta = - \sum_{i=m-}^{m+} \frac{1}{\Lambda} = -\frac{2(m+1)}{\Lambda} = -1. \qquad (16)$$

Likewise for the mean cycle squared jumping number,

$$\begin{aligned} \langle \hat{n}^2 \rangle_\zeta &= -\frac{(\hat{n}_{0+})^2}{\Lambda} - \cdots - \frac{(\hat{n}_{m+})^2}{\Lambda} - \frac{(\hat{n}_{m-})^2}{\Lambda} - \cdots - \frac{(\hat{n}_{0-})^2}{\Lambda} & (17) \\ &= -\frac{2}{\Lambda} \sum_{i=0}^{m} i^2 = -\frac{m(m+1)(2m+1)}{3\Lambda} = -\frac{(\Lambda-1)(\Lambda-2)}{12}. & (18) \end{aligned}$$

Plugging into the formula for the diffusion constant is trivial and gives

$$D = \frac{1}{2} \frac{\langle \hat{n}^2 \rangle_\zeta}{\langle n \rangle_\zeta} = \frac{(\Lambda-1)(\Lambda-2)}{24} \qquad (19)$$

Voila!

2 Sawtooth map, cut in (Markov) pieces

The preceding discussion was made simple because $\hat{f}$ mapped the $\mathcal{M}_i$ subintervals onto the entire unit interval I. This made the symbolic dynamics easy since we did not have to keep track of what had happened in the preceding steps. We could use the jumping numbers as alphabet and the dynamics was unrestricted.

However, we will now go a step further and consider situations where the unit interval can be partitioned into a finite number of subintervals $\mathcal{M}_i$ which only has the weaker property that; loosely speaking, subintervals are mapped onto unions of subintervals. This is more technically correct written as

$$f(\mathcal{M}_i) \cap \mathcal{M}_j = \emptyset \quad \text{or} \quad \mathcal{M}_j \subset f(\mathcal{M}_i) \qquad \forall i, j \qquad (20)$$

Such a partition is called *finite Markov.*

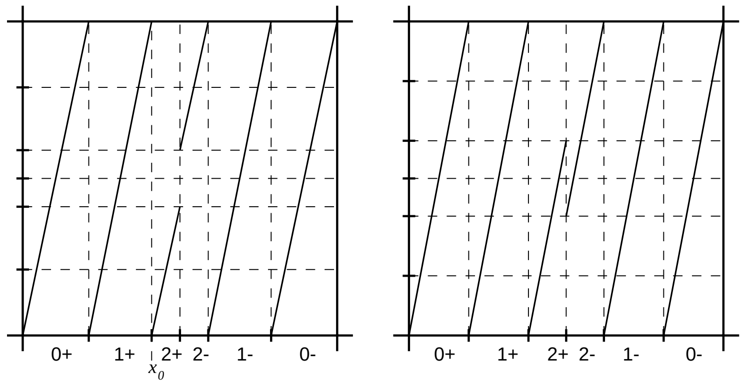

Figure 2: Illustration of the mapping $\hat{f}$ for $\Lambda = 2 + 2\sqrt{2}$ (left) and $\Lambda = 3 + \sqrt{5}$ (right). The jumping numbers for the intervals are indicated.

We will now consider something concrete, a map allowing a finite Markov partition $\{\mathcal{M}_{0+}, \mathcal{M}_{1+}, \mathcal{M}_{2+}, \mathcal{M}_{2-}, \mathcal{M}_{1-}, \mathcal{M}_{0-}\}$ where the (critical) point $1/2$ is mapped onto the right end-point of the interval $\mathcal{M}_{1+}$. The map is illustrated on figure 2.

First of all, we will determine the value of Λ that corresponds to this situation. What we know is that the critical point $1/2$ is mapped onto the right end-point of $\mathcal{M}_{1+}$, x_0. However, we know that being the right end-point of $\mathcal{M}_{1+}$ x_0 solves the equation $\hat{f}(x_0) = 2$, so that we have the following equations to solve for Λ,

$$f(1/2) = x_0, \text{ or } \ \hat{f}(1/2) = x_0 + 2 \ \text{ together with } \ \hat{f}(x_0) = 2. \tag{21}$$

First solving the last equation for x_0 gives $\Lambda/2$ and substituting into the first equation gives $\Lambda = 2(1 \pm \sqrt{2})$. The negative solution is obviously not the one we are seeking and we have already limited our scope to maps with $\Lambda > 2$ so it is discarted.

By inspecting figure 2 it is clear that $f(\mathcal{M}_i) = I$ for $i \in \{0+, 1+, 1-, 0-\}$, $f(\mathcal{M}_{2+}) = \mathcal{M}_{0+} \cup \mathcal{M}_{1+}$ and $f(\mathcal{M}_{2-}) = \mathcal{M}_{0-} \cup \mathcal{M}_{1-}$, so we can choose an alphabet $\mathcal{A}$ in which the symbolic dynamics is unrestricted, $\mathcal{A} = \{0+, 1+, 2+0+, 2+1+, 2-1-, 2-0-, 1-, 0-\}$.

And the zeta function is

$$1/\zeta = 1 - t_{0+} - t_{1+} - t_{2+0+} - t_{2+1+} - t_{2-1-} - t_{2-0-} - t_{1-} - t_{0-} \tag{22}$$

We can now grind out the numbers,

$$\langle n \rangle_\zeta = 4\frac{1}{\Lambda} + 4\frac{2}{\Lambda^2} = \frac{4\Lambda + 8}{\Lambda^2} \tag{23}$$

and

$$\langle \hat{n}^2 \rangle_\zeta = 2\frac{0^2}{\Lambda} + 2\frac{1^2}{\Lambda} + 2\frac{2^2}{\Lambda^2} + 2\frac{3^2}{\Lambda^2} = \frac{2\Lambda + 26}{\Lambda^2}, \tag{24}$$

and the diffusion constant is

$$D = \frac{1}{2}\frac{\langle \hat{n}^2 \rangle_\zeta}{\langle n \rangle_\zeta} = \frac{\Lambda + 13}{4\Lambda + 8} = \frac{15 + 2\sqrt{2}}{16 + 8\sqrt{2}} = \frac{26 - 11\sqrt{2}}{16}. \tag{25}$$

Voila!

The basic observation that made the previous calculation work was that we could find a partition which was Markov. The point 1/2 was mapped to the end of an interval and allowed a Markov partition to be found. We will now work out a few more examples of this sort for values of Λ between 4 and 6. There are 7 easy-to-find such examples, 1/2 mapped to the left end-points of $\mathcal{M}_{0+}$ ($\Lambda = 4$), $\mathcal{M}_{1+}$ ($\Lambda = 2+\sqrt{6}$), $\mathcal{M}_{2+}$ ($\Lambda = 2+2\sqrt{2}$), the right end-points of $\mathcal{M}_{2+}$ ($\Lambda = 5$), $\mathcal{M}_{2-}$ ($\Lambda = 3+\sqrt{5}$), $\mathcal{M}_{1-}$ ($\Lambda = (5+\sqrt{41})/2$) and $\mathcal{M}_{0-}$ ($\Lambda = 6$).

Some of the numbers we already have worked out. The integers are covered by the results from the previous section. We find from the formulas that $D = 1/4$ for $\Lambda = 4$, $D = 1$ for $\Lambda = 5$ and $D = 5/6$ for $\Lambda = 6$. The case $\Lambda = 2+2\sqrt{2}$ was calculated above and we quote the result $D = (26 - 11\sqrt{2})/16$.

The case $\Lambda = 2+\sqrt{6}$ can be dealt with along lines similar to the $\Lambda = 2+2\sqrt{2}$ case. The dynamics is unrestricted in the alphabet $\mathcal{A} = \{0+, 1+, 2+0+, 2-0-, 1-, 0-\}$, so we can write

$$1/\zeta = 1 - t_{0+} - t_{1+} - t_{2+0+} - t_{2-0-} - t_{1-} - t_{0-} \tag{26}$$

to calculate

$$\langle \hat{n}^2 \rangle_\zeta = -\frac{2}{\Lambda} - \frac{8}{\Lambda^2} = -\frac{2\Lambda + 8}{\Lambda^2} \tag{27}$$

and

$$\langle n \rangle_\zeta = -\frac{4}{\Lambda} - \frac{4}{\Lambda^2} = -\frac{4\Lambda + 4}{\Lambda^2}. \tag{28}$$

This gives us immediately the diffusion constant

$$D = \frac{1}{2}\frac{\langle \hat{n}^2 \rangle_\zeta}{\langle n \rangle_\zeta} = \frac{\Lambda + 4}{4\Lambda + 4} = \frac{6+\sqrt{6}}{12+4\sqrt{6}} = 1 - \frac{\sqrt{6}}{4} \tag{29}$$

The Duke of Cambridge

The two remaining cases are a little harder but with a little persistence we will be able to deal with them. Let us first look at the case of $\Lambda = 3+\sqrt{5}$.

The unit interval is naturally partitioned into 6 subintervals as in the preceding examples. The intervals with labels 0+, 1+, 1− and 0− are mapped onto the entire unit interval. What gives a little trouble is the two middle intervals, $\mathcal{M}_{2+}$ and $\mathcal{M}_{2-}$. The first of these two is mapped onto the intervals $\mathcal{M}_{0+}$, $\mathcal{M}_{1+}$, $\mathcal{M}_{2+}$ and $\mathcal{M}_{2-}$, and the second is mapped onto $\mathcal{M}_{0-}$, $\mathcal{M}_{1-}$, $\mathcal{M}_{2-}$ and $\mathcal{M}_{2+}$. Writing down the alphabet corresponding to the dynamics is a bit tricky. It will have to contain sequences of arbitrarily long combinations of 2+ and 2− terminating with one of the four symbols which gives no restriction on the next letter, 0+, 1+, 1− and 0−. We will, however, have to make sure that the label before the terminating label in the letter has the right sign, e.g. before 1− we will have to have 2− and not 2+. On top of this it will have to have provisions for itineraries which let the point bounce back and forth between $\mathcal{M}_{2+}$ and $\mathcal{M}_{2-}$ indefinitely.

However, we can avoid the problems of formulating the alphabet directly and make use of the fact that the slope of the map is the constant Λ. The latter gives rise to identities of the form $t_{(2+)^{k_1}(2-)^{k_2}(2+)^{k_3}0+} = (t_{2+})^{k_1+k_3}(t_{2-})^{k_2}t_{0+}$

etc. as can be seen directly from the definition of t_p. These identities make shadowing work and they will make life much easier in the following.

The sequences of labels which are supposed to make out the alphabet can be of three forms. It can be one of 0+, 1+, 1− and 1− in which case there are no problems what so ever. There can be a sequence — possibly empty — of 2+s and 2−s preceding a $2+0+$, $2+1+$, $2-1-$ or $2-0-$ as described above. It is important to note that the order of the symbols matter when the alphabet is written out but when the zeta function is calculated we can use the identities noted above. The problem of ordering is thus reduced to one of counting. Postponing the third case for a moment, we write

$$\begin{aligned}(1 \; - \; & t_{0+} - t_{1+} - t_{1-} - t_{0-} \\ - \; & (t_{2+}t_{0+} + t_{2+}t_{1-} + t_{2-}t_{1-} + t_{2-}t_{0-}) \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} (t_{2+})^i (t_{2-})^{n-i}).\end{aligned} \tag{30}$$

We have here introduced the binominal coefficient $\binom{n}{i}$ to count the number of ways i plus-signs can be distributed over n symbols. The factor in front of the sum takes care of the fact that the symbol sequences has to end with the correct combinations of $2\pm$ and $0\pm/1\pm$. Shadowing accounts for the fact that we don't get an infinite product to work out but only a sum.

We will now return to the third case we left out before. It is the case where the sequences consists only of $2\pm$. This is analogous to the case of Λ odd where we had to include the two fixed points. We deal with it in rather much the same way by introducing them into to the infinite product by hand. However, here there are infinitely may possible sequences whereas the two fixed points just gave two additional factors. Fear not, dear reader, yet again one can hear the cavalary approaching — shadowing sets in and cancles the curvature corrections exactly, leaving only the term

$$(1 - t_{2+} - t_{2-}). \tag{31}$$

We can now write out the full dynamical zeta function, however, it pays of to give it a bit of massage. The sums in (30) can easily be calculated by noting that

$$\sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} (t_{2+})^i (t_{2-})^{n-i} = \sum_{n=0}^{\infty} (t_{2+} + t_{2-})^n = \frac{1}{(1 - t_{2+} - t_{2-})}. \tag{32}$$

The oracle can now be calculated,

$$1/\zeta = (1 - t_{2+} - t_{2-})(1 - t_{0+} - t_{1+} - t_{1-} - t_{0+} - \frac{(t_{2+}t_{0+} + t_{2+}t_{1+} + t_{2-}t_{1-} + t_{2-}t_{0-})}{(1 - t_{2+} - t_{2-})}). \tag{33}$$

Multiplying everything out and collecting terms gives us

$$1/\zeta = 1 - t_{0+} - t_{1+} - t_{2+} - t_{2-} - t_{1-} - t_{0-} + t_{2+}t_{0-} + t_{2+}t_{1-} + t_{2-}t_{0+} + t_{2-}t_{1+}. \tag{34}$$

Calculating the mean cycle square jumping number and mean cycle length is now trivial,

$$\langle \hat{n}^2 \rangle_\zeta \; = \; -t_{1+} - 4t_{2+} - 4t_{2-} - t_{1-} + 4t_{2+}t_{0-} + t_{2+}t_{1-} + t_{2-}t_{1+} + 4t_{2-}t_{0+}$$

$$
\begin{aligned}
&= -10\frac{\Lambda-1}{\Lambda^2} && (35)\\
\langle n\rangle_\zeta &= -t_{0+}-t_{1+}-t_{2+}-t_{2-}-t_{1-}-t_{0-}+2t_{2+}t_{0-}+2t_{2+}t_{1-}+2t_{2-}t_{1+}+2t_{2-}t_{0+}\\
&= -\frac{6\Lambda-8}{\Lambda^2}. && (36)
\end{aligned}
$$

The diffusion constant D follows,

$$D = \frac{1}{2}\frac{\langle \hat{n}^2\rangle_\zeta}{\langle n\rangle_\zeta} = \frac{5\Lambda-5}{6\Lambda-8} = \frac{5+\sqrt{5}}{8}. \tag{37}$$

The case $\Lambda = (5+\sqrt{41})/2$ can be dealt with along similar lines. The main difference is that now the intervals $\mathcal{M}_{2+}$ and $\mathcal{M}_{2-}$ are mapped onto $\mathcal{M}_{0+}\cup\mathcal{M}_{1+}\cup\mathcal{M}_{2+}\cup\mathcal{M}_{2-}\cup\mathcal{M}_{1-}$ and $\mathcal{M}_{0-}\cup\mathcal{M}_{1-}\cup\mathcal{M}_{2-}\cup\mathcal{M}_{2+}\cup\mathcal{M}_{1+}$, respectively. This makes only a small difference in the calculations since we only need to take into account the fact that combinations of $2+$ and $2-$ now also can terminate with $2+1-$ and $2-1+$. This is done by introducing $t_{2+}t_{1-}$ and $t_{2-}t_{1+}$ in the factor in front the sums in eq. (30). Doing the maths analogous to what we did before gives

$$1/\zeta = 1 - t_{0+} - t_{1+} - t_{2+} - t_{2-} - t_{1-} - t_{0-} + t_{2+}t_{0-} + t_{2-}t_{0+}. \tag{38}$$

It is interesting to notice the way terms with two t_p-s cancel when we make the image of $\mathcal{M}_{2+}$ overlap all but the last interval, $\mathcal{M}_{0-}$. It is fairly obvious what will happen when we let the interval overlap all the subinterval, i.e. we consider the case of $\Lambda = 6$; we get exactly the simple form of the zeta function corresponding to unrestricted dynamics in the alphabet $\{0+, 1+, 2+, 2-, 1-, 0-\}$. It is reassuring to know that our methods are at least consistent!

Applying the operators $z\frac{\partial}{\partial z}$ and $\frac{\partial^2}{\partial\beta^2}$ and evaluating at $(\beta, z) = (0,1)$ gives

$$\langle \hat{n}^2\rangle_\zeta = -\frac{10\Lambda-8}{\Lambda^2} \quad\text{and}\quad \langle n\rangle_\zeta = -\frac{6\Lambda-4}{\Lambda^2}. \tag{39}$$

The diffusion constant now follows readily,

$$D = \frac{1}{2}\frac{\langle \hat{n}^2\rangle_\zeta}{\langle n\rangle_\zeta} = \frac{5\Lambda-4}{6\Lambda-4} = \frac{107-\sqrt{41}}{124}. \tag{40}$$

If we compare with table H.2 in the project description we see that the results calculated here are different from those in the table for (a), (b) and (e). The two first are readily explained as typos in the table(?) but it seem very possible that a small error found its way into my calculations of the second last entry in the table.

3 Numerics and some concluding remarks

When one has no other ideas one can always try to do some numerical calculations... We have now calculated a few diffusion constants, trusting that we have derived the underlying theory correctly. If we can trust periodic orbit theory then our calculations of diffusion constants have been exact. However, it is not intuitively obvious that the numbers found so far are correct so we might want

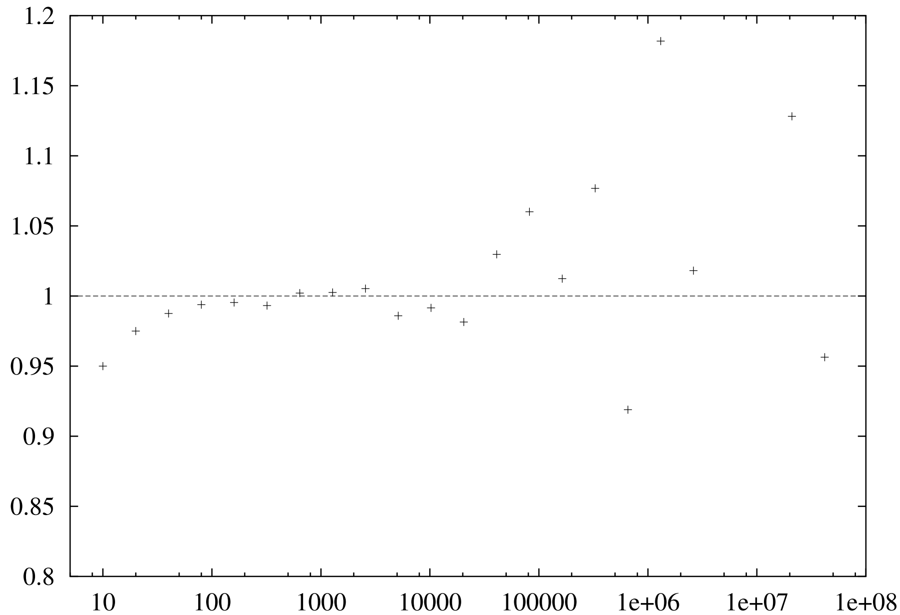

Figure 3: Diffusion constant calculated numerically for $\Lambda = 5$ with increasing numbers of iterations. The value $D = 1$ found from periodic orbit theory is indicated with a dashed line.

to use the dumb approach of brute force numerics. It is a quite obvious approach since iterated mapping lend themselves to easy computer implementation.

However, contrary to popular belief, numerical calculations is a quite subtle subject and the pitfalls are *legio*. We will here look at data for the case of $\Lambda = 5$ calculated for increasing numbers of iterations. Because we do not have infinitely much computer-time, we will balance the number of iterations with the number of start-points we look at. Figure 3 shows the diffusion constants as a function of the number of iterations in a simulation where the number of iterations times the number of input points has been kept at 10^8. (This takes about 70 s per data-point on a fast computer with an extremely inelegant C-program using double precision arithmetics.) The result, $D = 1$, from periodic orbit theory is also indicated on the figure. It is obvious that the result from periodic orbit theory is of the right order of magnitude but something is clearly going wrong.

From the figure we see that up to a few thousand iterations the points look as if they are converging but the they seem to be tossed around more or less at random for more iterations. What seems to be happening is that the sequence of points gets caught by an attractor for the dynamical system consisting of both the iterated mapping and non-linearities caused by round-off errors. This gives rise to apparently extreme diffusion constants.

Λ	*diffusion constant*	
4	$\frac{1}{4}$	0.25
$2+\sqrt{6}$	$1-\frac{\sqrt{6}}{4}$	≈ 0.388
$2+2\sqrt{2}$	$\frac{26-11\sqrt{2}}{16}$	≈ 0.653
5	1	1
$3+\sqrt{5}$	$\frac{5+\sqrt{5}}{8}$	≈ 0.905
$\frac{5+\sqrt{41}}{2}$	$\frac{107-\sqrt{41}}{124}$	≈ 0.811
6	$\frac{5}{6}$	≈ 0.833

Where did we end up, and where do we go from here?

But what about the exact results, then? Can we interpret them? We can see that the largest value of D is found for $\Lambda = 5$. Comparing figure 1 this might be possible to understand. First of all, we would expect as a general trend that higher values of Λ would lead to faster diffusion. It is true that diffusion gets faster with higher values of Λ. The expressions for Λ even and odd also shows that: for large values D goes as Λ^2. This is, however, not the only effect so can we understand why $\Lambda = 5$ gives the highest value of D.

Every time the sequence enters the $\mathcal{M}_{2+}$ interval it will not only be sent far in one direction, it will also be sent in the same direction in the next time step as can be seen from the figure (by symmetry this is also true for $2-$). If we increase Λ to $3+\sqrt{5}$ we get the possibility that the 'fast running' points of $\mathcal{M}_{2+}$ get mapped to the $\mathcal{M}_{2-}$ interval and thus kicked in the other direction. If we then increase Λ one step more we get a further possibilities to get sent in the opposite direction when we make a long jump and thus a lower value of the diffusion constant even though Λ goes up. If we go to $\Lambda = 6$, points in $\mathcal{M}_{2+}$ can now be mapped to $\mathcal{M}_{0-}$ also but that just means standing still for one iteration so we would expect this to affect the diffusion constant in a less negative way and indeed it even does go up a bit.

These are of course just hand-waving arguments but they seem to explain the behaviour seen in the calculated values of the diffusion constant. What one should be able to work out is whether there is a general trend of D growing roughly as Λ^2 and then eliminate that effect and somehow study the 'bare' effect of letting the 'fastest running' intervals, $\mathcal{M}_{m\pm}$, overlap more or less with the other intervals. It also seems possible that the effects of the overlaps of the $\mathcal{M}_{m\pm}$ intervals and the other intervals will diminish as we go to higher values of Λ. It might happen that there exists a point from which the diffusion constant grows 'monotonically' with Λ increasing in steps that match Markov partitions of the simple form we have studied here (e.g. finite Markov partitions labeled by jumping numbers). If this actually happens one can study whether something similar will happen if we make even finer partitions (a possibility not even mentioned here), and one can maybe investigate whether from some finite value onwards D becomes a monotonic function of Λ (my guess is that no such value exists). All these are possibilities which can be investigated, possibly with techniques used in this term paper.