Looking for creeping complex-orbits in a radially symmetric potential.

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I. INTRODUCTION

When solving classical scattering problems, a particle is deflected from its initial trajectory by certain potential (scatterer) and one calculates the scattering angle, the flux that makes its way through an obstacle (a colection of scatterers for example) or evaluate the periodic orbits that occur, with which we may evaluate the previously mentioned quantities, and much more. The problem with this classical approach is that it does not consider the wave nature of the particle, and therefore it does not take into account effects like creeping of the particle around corners.

The analytic calculation of complex-orbits that creep around solid wall potentials (infinite potential) has been already worked for a $2D$-circular infinite potential [1]. The methods used to calculate the different periodic orbits in this kind of potential, depend highly in the specific problem to be solved and are usually applicable to a handful of special cases, where the symmetry is high. It is desirable to be able to calculate these complex orbits in a more general way, because this would let us automatize the process of finding the orbits as long as we are able to define the equations of motion for the system.

The idea here is to take a very simple $2D$-potential, derive the Hamilton-Jacobi equations for this system and then propose that both the spacial and momenta variables are complex. We will then have a new system of equations in which we may find the complex orbits we are looking for.

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II. BASIC EQUATIONS

We will work with a simple $2D$ system of a free-particle moving in the presence of a Gaussian potential,

$$\begin{aligned} T(\dot{\vec{r}}) &= \frac{1}{2}m|\dot{\vec{r}}|^2, \\ V(\vec{r}) &= V_0 e^{-\frac{1}{2d^2}|\vec{r}|^2} \end{aligned} \tag{1}$$

We can easily identify de Lagrangian of the system to be

$$L(\vec{r},\dot{\vec{r}}) = \frac{1}{2}m|\dot{\vec{r}}|^2 - V_0 e^{-\frac{1}{2d^2}|\vec{r}|^2}. \tag{2}$$

With the Lagrangian we may calculate the generalized momentum, which is the usual

$$\frac{\partial L}{\partial \dot{r_j}} = m\dot{r_j} = p_j, \tag{3}$$

where the index j depends on the number of dimentions of the problem. Finally we may write the Hamiltonian as

$$H(\vec{r},\vec{p}) = \frac{|\vec{p}|^2}{2m} + V_0 e^{-\frac{1}{2d^2}|\vec{r}|^2}. \tag{4}$$

The Hamilton-Jacobi equations are given by

$$\begin{aligned} \dot{r_j} &= \frac{\partial H}{\partial p_j} = \frac{p_j}{m}, \\ \dot{p_j} &= -\frac{\partial H}{\partial r_j} = \frac{V_0 r_j}{d^2} e^{-\frac{1}{2d^2}|\vec{r}|^2}. \end{aligned} \tag{5}$$

Its at this point where we say that we will allow both, position and momentum to be complex, so we will have $r_j \to x_j + iy_j$ and $p_j \to \pi_j + i\xi_j$. The new set of Hamilton-Jacobi equations, in terms of the x, y, π and ξ variables would be

$$\dot{x_j} = \frac{\pi_j}{m}$$
$$\dot{y_j} = \frac{\xi_j}{m}$$
$$\dot{\pi_j} = V_0 e^{-\frac{1}{2d^2}\sum_{j'}(x_{j'}^2 - y_{j'}^2)} \left[x_j \cos\left(\frac{1}{d^2}\sum_{j'} x_{j'} y_{j'}\right) + y_j \sin\left(\frac{1}{d^2}\sum_{j'} x_{j'} y_{j'}\right)\right]$$
$$\dot{\xi_j} = V_0 e^{-\frac{1}{2d^2}\sum_{j'}(x_{j'}^2 - y_{j'}^2)} \left[y_j \cos\left(\frac{1}{d^2}\sum_{j'} x_{j'} y_{j'}\right) - x_j \sin\left(\frac{1}{d^2}\sum_{j'} x_{j'} y_{j'}\right)\right]. \quad (6)$$

With this system of equations we may now look for complex periodic orbits.

III. COMPUTATIONAL APPROACH

Once the system of differential equations is defined, the next step is to integrate the system to obtain trajectories in the complex phase space. The ultimate goal is to find closed trajectories, which will constitute the complex periodic orbits.

After being able to integrate the system, in this case with a Runge-Kutta algoritm, we need to teach our program to find the periodic orbits we are interested in. To achieve this, we use a multi-point shooting method [1]. In a multi-point shooting method, its necesary to have several guesses for points of the orbit, and the algoritm, through a series of Newton-Rapson-like steps, evolves these initial conditions to the actual periodic orbit.

A. First Glance at the Orbits

In general, its a good idea to gather some information about the particle's trajectories before starting to feed the computer with guesses. We know what to expect for the real Hamiltonian's case, since this potential constitutes a radially symmetric scatterer. After we complexify the equations, we can be no longer sure about the behaviour of the orbits and the effect of having complex components for the position and momenta.

The first thing that was done was to explore the types of orbits that would be obtained for different initial conditions. We expect to find a periodic orbit that is concentric with the potential, due to the symmetry of the potential in use. We also expect that the real momentum will be perpendicular to the real trajectory, so, we can have a general idea of the types of orbits we may

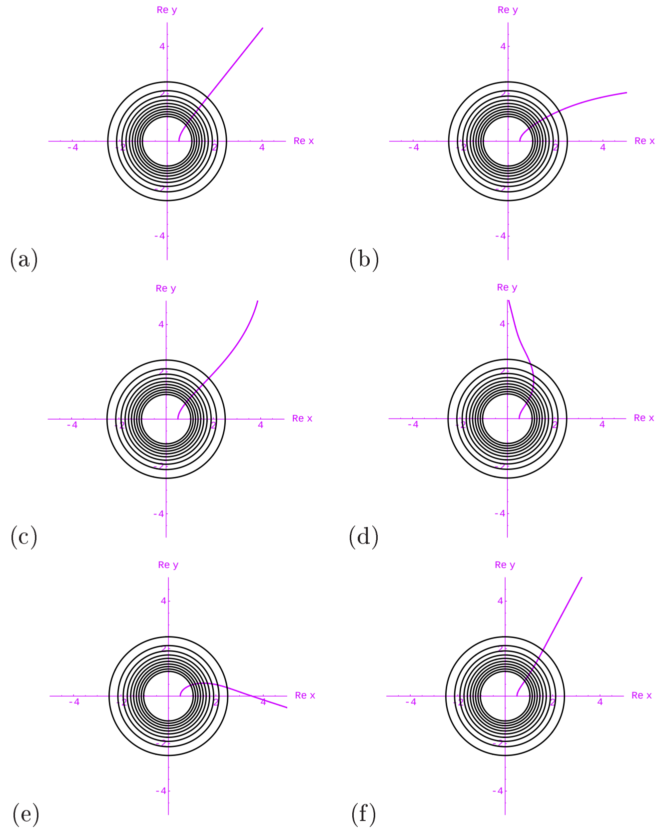

FIG. 1: (a)-(f) Different initial conditions for complex position and momenta. $m = 1$, $d = 1$, $V_0 = 1$.

obtain, by placing the particle at different positions on the real r_1 axis, with real r_2 component zero, and with real momentum p_2 positive, keeping the real part of the momentum p_1 equal to zero. We leave the complex components of both position and momenta as parameters, and for a given position in r_1, we explore several configurations of these complex values, and compute the orbits they produce.

The conditions for the orbits shown in the figure [1] are:

$$\begin{array}{llll}
(a) & \overrightarrow{r_0} = (0.5, 0) & , & \overrightarrow{p_0} = (0, 0.5) \\
(b) & \overrightarrow{r_0} = (0.5 - i0.5, i0.5) & , & \overrightarrow{p_0} = (0, 0.5) \\
(c) & \overrightarrow{r_0} = (0.5, 0) & , & \overrightarrow{p_0} = (-i0.5, 0.5 + i0.5) \\
(d) & \overrightarrow{r_0} = (0.5, -i0.5) & , & \overrightarrow{p_0} = (i0.5, 0.5 - i0.5) \\
(e) & \overrightarrow{r_0} = (0.5 + i0.5, -i0.5) & , & \overrightarrow{p_0} = (i0.5, 0.5) \\
(f) & \overrightarrow{r_0} = (0.5 + i0.5, i0.5) & , & \overrightarrow{p_0} = (0, 0.5 + i0.5)
\end{array}$$

In figure [1.a] the classical situation is shown, the motion occurs, as expected, repelling from the central potential. In the other cases shown, there are various effects on the orbit due to the imaginary components of the position and momenta. The energies are not the same in all the cases shown.

The next thing that was checked was the change of the shape of the orbit with a change on the real r_1 position.

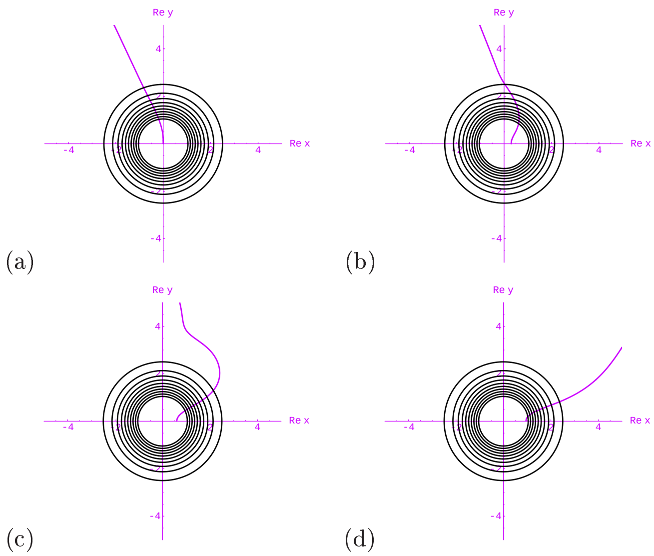

FIG. 2: (a)-(d) Changing the initial real r_1 from 0 to 0.95. $m = 1$, $d = 1$, $V_0 = 1$.

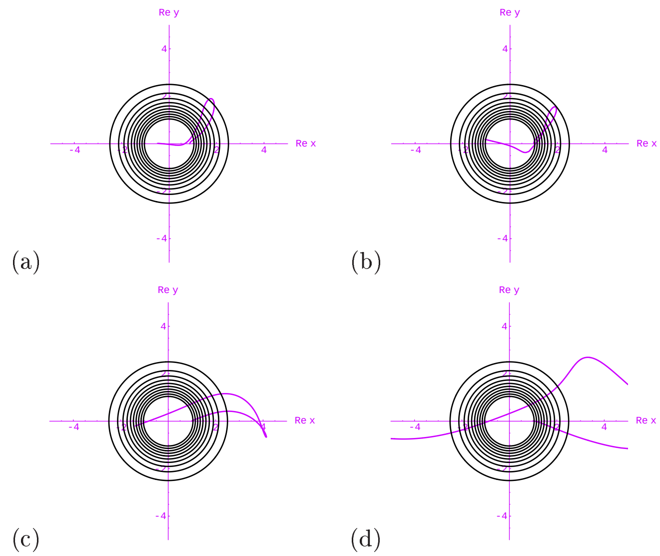

FIG. 3: (a)-(d) Different orbits that visit the four real space quadrants.

For this one of the initial conditions of the previus set that yielded an orbit that bent to the second quadrant was chosen, and the real part of r_1's initial condition was varied.

The next step was to check whether there were orbits that would visit all four quadrants or not, to verify this, several initial conditions were integrated, and only r_2 and p_1's real parts were kept at zero value. The other variables were varied and several hundreds of thousands of trajectories were screened. The outcome was not the expected though, some of the orbits effectibly visited the four real space quadrants, but in none of the cases an orbit that would seem to approximate a periodic orbit was observed.

A curious fact is that for every single of the orbits that visit the four quadrants the real part of the energy yielded a negative value, which could suggest that negative real energies are necesary for bounded motion in this kind of potential to occur.

B. Difficulties

The first problem we face is the selection of reasonable initial conditions. As it can be shown, if we start with complex initial conditions, the energy will be complex (in general) and therefore, due to the conservation of energy inherent to the Hamilton-Jacobi equations we will face the problem of trying to link this complex energy with the real energy of the incomming particle. When we consider the physical problem at hand we notice that the incomming particle must have a real energy, since for large distances to the scatterer the particle must behave as a free particle. Only in the vecinity of the potential should the particle's motion be affected by it.

A possible solution to the problem of linking the real and complex energies is considering what happens when the particle approaches the potential wall. It is possible to think of it as if the time went complex when the radial momentum is close to zero, in this way we should change our Hamilton-Jacobi equations using

$$\frac{d}{dt} \to -i\frac{d}{d\tau}, \tag{7}$$

which basically changes our Hamilton-Jacobi equations to

$$\begin{aligned}
\dot{y_j} &= \frac{\pi_j}{m} \\
\dot{x_j} &= -\frac{\xi_j}{m} \\
\dot{\xi_j} &= V_0 e^{-\frac{1}{2d^2}\sum_{j'}(x_{j'}^2 - y_{j'}^2)}\left[x_j \cos\left(\frac{1}{d^2}\sum_{j'} x_{j'} y_{j'}\right)\right. \\
&+ \left. y_j \sin\left(\frac{1}{d^2}\sum_{j'} x_{j'} y_{j'}\right)\right] \\
\dot{\pi_j} &= -V_0 e^{-\frac{1}{2d^2}\sum_{j'}(x_{j'}^2 - y_{j'}^2)}\left[y_j \cos\left(\frac{1}{d^2}\sum_{j'} x_{j'} y_{j'}\right)\right. \\
&- \left. x_j \sin\left(\frac{1}{d^2}\sum_{j'} x_{j'} y_{j'}\right)\right].
\end{aligned} \tag{8}$$

When we do this we are calculating what happens when the particle tunnels through the potential. When this is done in $1D$, what happens is that we basically solve the same potetial but now upside-down. A similar situation would occur in the $2D$ case, but what will happen is that we will have several complex periodic orbits with complex time, all of which should contribute to the measured final effects.

This part has not been worked numerically yet, but seems to be a good way to explain how to obtain complex energies when you start with a purelly real energy (free particle case), so at least the conceptual concern we had before would be solved if we could connect the change to complex time with the complexification of the energy to obtain bounded orbits.

IV. DISCUSSION

Overall, the final goal was not achieved. Several conceptual problems are still unanswered, and the solution of the problem does not look clear. Yet, some clearer understanding of the interpretation of the complex-orbits has been gained.

It is still important to perform the numerical calculations making the time complex, to see if the analogy complex time - tunneling also holds for $2D$ potentials.

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[1] P. Cvitanović *et al.*, *Chaos: Classical and Quantum*, advanced graduate e-textbook, available online at `ChaosBook.org` (Niels Bohr Institute, Copenhagen 2003).