# Studying a Driven Pendulum with a $1D$-circle-map

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A method to model a $2D$-Poincaré-map of a driven pendulum with a $1D$-circle-map is proposed. The symbolic dynamics for the system is specified as well as the pruning rules. A method to calculate the periodic orbits is proposed and implemented. The averages calculated with the model are $\cdots$ compared to the original system.

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**Georgia Tech PHYS 7123:**
**CHAOS, AND WHAT TO DO ABOUT IT**
**course project, fall semester 2003**

## I. INTRODUCTION

I am working on a project proposed to me by Dr. Slaven Peleš[1]. The idea is to attempt to model a $2D$-Poincaré-map of a driven pendulum with a $1D$-*circle-map*, in order to predict (roughly) the averages for the original system.

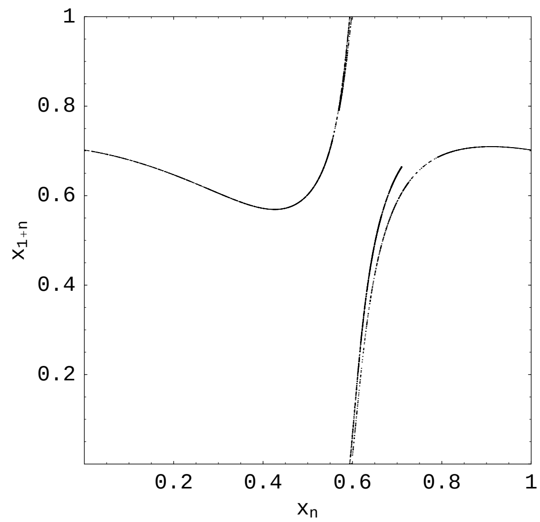


FIG. 1: Poincaré section of a driven pendulum system, $2D$ map

The approach consists in analyzing the symbolic dynamics of the model $1D$ map, defining the prunning rules and determining numerically the shorter cycles, implementing the $1D$-model periodic-orbit calculations, and then try to extrapolate them back to the original system. The motivation for trying to use a $1D$-map is simple: for the $1D$-map one can calculate things exactly, and the dynamics of circle maps are well studied in the literature[2, 3].

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## II. BASIC EQUATIONS

We will model the $2D$-Poincaré map (Fig. 1) with a $1D$-circle-map of the form

$$f(x_n) = x_n + a - \frac{b}{2\pi}\sin(2\pi x_n + \delta) \mod 1, \tag{1}$$

where $a$, $b$ and $\delta$ are parameters of the fit. The function $f(x)$ maps the $[0, 1]$ interval into itself. For $a < 1$ and $b > 1$, the map has a local minimum and a local maximum as shown in Fig.2, so we can model the original system with the right choiceof the parameters. The parameters can be chosen by adding conditions of the form

$$\begin{aligned} f(x_a) &= \alpha, \text{or} \\ f'(x_a) &= \beta \end{aligned}$$

and solving the resulting system of equations (constraints) for the parameters.

### A. Parameters of the fit

When looking for the optimal parameters for the fit(1) one may attempt to add constraints and solve a system of equations. The problem is that usually one obtains similarly good fits by doing educated guesses on the parameters by observing the function itself.

Looking for a good fit for the original system, I found it convenient to use a circle map with two harmonic terms with different frequencies, so instead using a circle map of the form (1)I am going to use a circle map of the form

$$\begin{aligned} x_{n+1} &= \Big[x_n + a - \frac{b}{2\pi}\sin(2\pi x_n + \delta) \\ &- \frac{c}{4\pi}\sin(4\pi x_n + \gamma)\Big] \mod 1. \end{aligned} \tag{2}$$

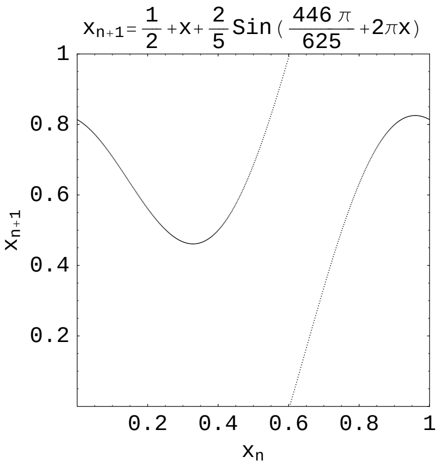


FIG. 2: Circle map, $a = 0.5$, $b = 2.51327$ and $\delta = 2.24184$

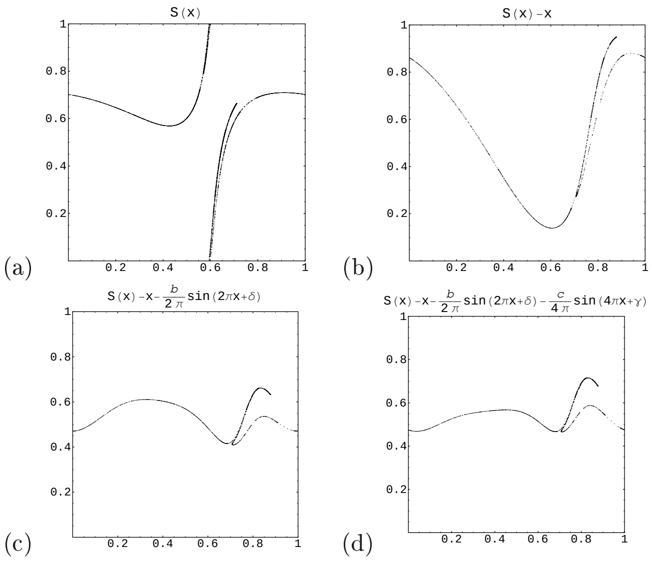


FIG. 3: (a)-(d) Process of narrowing the fit.

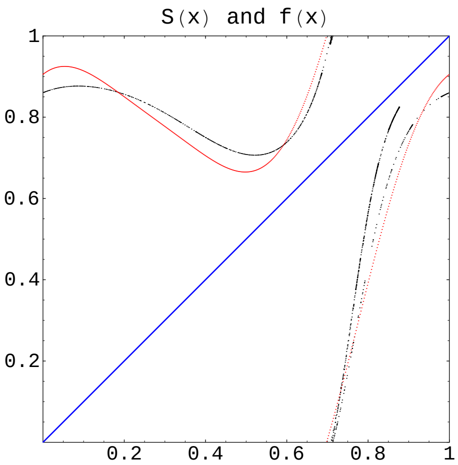


FIG. 4: Fit with parameters $a = 0.52$, $b = -2.3562$, $c = -1.131$, $\delta = 11\pi/20$, $\gamma = 9\pi/10$

or more terms in the Fourier expansion.

The approach followed to obtain the final fit, $f(x)$, was to attempt to cancel the original data (make it go to zero) by substracting the correction terms. The sum of all the correction terms gives us $f(x)$.

There are two important properties on the circle map,

$$\begin{aligned} f(1) &= f(0) \quad \text{and,} \\ f^{(n)}(1) &= f^{(n)}(0) \end{aligned}$$

which ensure continuity on the function and its derivatives.

In order to maintain these properties, we must restrict limit ourselves to choosing a coefficient 1 for the linear term, and restrict the inclusion of harmonic terms to those with the general form $A\sin(2n\pi x + \epsilon)$.

The sequence in Fig.3 shows the process to obtain the fit, first substracting the linear term, then substracting the first harmonic term with an amplitude equal to half of the amplitude of the previous and an initial displacement was chosen such that the mean square difference between the fit and the original system was minimum for the already choosen amplitude. Then the second harmonic term is added, following the same prescription than that followed for the first and at last a constant is substracted to complete the fit, plotted in Fig.4.

Now the fit is ready, I can start coding the symbolic dynamics for this fit. One small issue still to be considered is the fact that the fits first maxima lies too close to $x = 0$, which could make things difficult to calculate (ie. a very small portion of the $[0, 1]$ interval will be mapped into this region before the first maxima).

## III. SYMBOLIC DYNAMICS

It is possible to partition the unit interval into *meaningful* intervals, by considering closely the extremal points and the discontinuities of the mapping function. It is convenient to have one symbol to represent each of the areas dictated by these points and we will follow the approach of [2, 4]. We will assign a symbol $M$ to those points $x_{n+1} = f(x_n)$ such that $0 < x_n < x_l$, where $x_l$ is the value of $x$ for which $f$ has its local minimum. In a similar way let us use $L$ for the points $x_{n+1} = f(x_n)$ such that $x_l < x_n < x_d$, where $x_d$ is the value of $x$ for which the discontinuity occurs. We then assign the symbols $R$ and $N$ to those $x_{n+1}$ lying in the regions $x_d < x_n < x_h$ and $x_h < x_n < 1$ respectibly, where $x_h$ is the value of $x$ for which $f$ has its local maxima. We will assume the order of the symbols based on the order of their occurrence on the unit interval. Also, it is important to notice that there are two special regions of our map, the ones represented by $M$ and by $L$ [6]. This regions are such that its slopes are negative, and it is easy to see that for two initial conditions lying in their interval of domain the order of their futures will switch after passing through it. In terms of these two properties, and by using the symbols for the regions of the map we may attempt to describe itineraries of the system[7] by their symbolic sequence, therefore greatly simplifying the information we need to describe said itineraries.

### A. Ordering Rules

In the previous section, the issue of the order of the symbolic sequences was addressed, and the importance of having a sense of order in our symbolic sequences will be clear after a few more paragraphs. The sense of order, along with the concept of periodic orbits, allows us to determine regions of the unit interval that may have certain symbolic future, actually, if the partition of the phase space was choosen wisely, it is probable that the sequences sharing a symbolic future will lie in a neighborhood in the phase space (unit interval). It will also permit us to determine, without too much effort, whether a proposed periodic orbit (in the symbolic sense) is a valir one or not, and even more, the fact that these neighborhoods tend to shrink when the itinerary increases in length, suggests that for a periodic orbit[8] this property will give us an exact value on the phase space for which the orbit occurs [9].

We can now attempt to define the ordering rules that in turn allow us to sort different itineraries,

1. $0 < M < x_l < L < x_d < R < x_h < N < 1$
2. For two different itineraries $I_a$ and $I_b$ with identical leading sequences $I$, $I_a = I\sigma_a \ldots$ and $I_a = I\sigma_b \ldots$, if $\sigma_a > \sigma_b$ then $I_a > I_b$ if the itinerary $I$ contains an even number of symbols $M$ and $N$'s and $I_a < I_b$ if the number of $M$ and $N$'s in $I$ is odd, this accounts for the order switching discussed before.

### B. Pruning Rules

Once we have a symbolic dynamics set, we may be tempted to think that we can now just come up with a random sequence of the symbols that constitute our alphabet and then just look for the points which correspond to this random sequence. The reality is different and we will see that the allowed sequences (words) will follow certain rules which will allow us to prune, out of all the possible words, those words that do not follow the prescribed rules[10]. This also is expected, since we mentioned above that a certain itinerary can be produced by a definite neighborhood, and that this neighborhood shrinks as the length of the itinerary is made longer. If all sequences were allowed it would mean that either we did not do a good job choosing the partition of our phase space or that the neighborhoods will not shrink as desired.

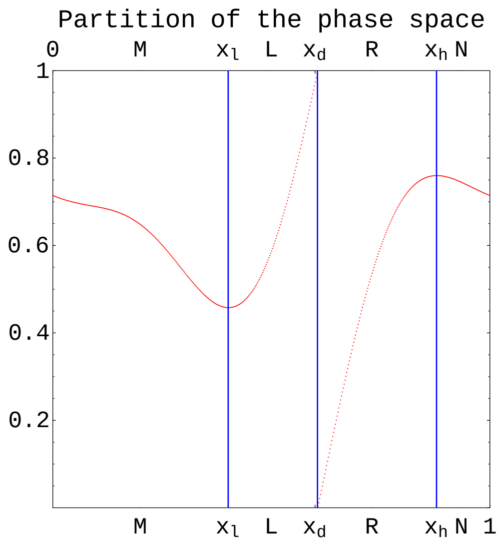


FIG. 5: Proposed partition of the phase space

To define the pruning rules let us first define a few terms in the sense they are defined in [2, 4]. Let $\mathbf{M}(I)$ be the set of all sequences of $I$ that lie immediately right of a letter $M$, and in the same sense define $\mathbf{L}(I)$, $\mathbf{R}(I)$ and $\mathbf{N}(I)$

$$\begin{aligned}
\mathbf{M}(I) &= \{\Lambda | I = \Gamma M \Lambda\} \\
\mathbf{L}(I) &= \{\Lambda | I = \Gamma L \Lambda\} \\
\mathbf{R}(I) &= \{\Lambda | I = \Gamma R \Lambda\} \\
\mathbf{N}(I) &= \{\Lambda | I = \Gamma N \Lambda\}. \qquad (3)
\end{aligned}$$

Let the special itineraries $\Phi_l$, $\Phi_h$ and $\Phi_0$(or $\Phi_1$) represent the itineraries of the minimal, maximal and extreme points ($x_l$, $x_h$ and 0 or 1). Then, in terms of the ordering rules and some immediate properties of the extrema itineraries we can write some restrictions, which are our pruning rules.

Notice first that the itinerary of any point $x_n$ whose image lies on $L$ (we could refeer to an itinerary on $\mathbf{L}$ as well) is always greater than the *minimal itinerary* $\Phi_l$, for $L$ is monotonically increasing. Also, any itinerary of $\mathbf{R}$ will be bounded by the *maximal* itinerary (these names will become clear in the next line) such that $\mathbf{R} \leq \Phi_h$ and at last we have that $\Phi_l \leq \mathbf{M} \leq \Phi_0$ and $\Phi_0 \leq \mathbf{N} \leq \Phi_h$.

So, the ordering rules and the pruning rules give us a way to construct all the possible existing periodic orbits for the system, and since the only calculation that we really need to do to obtain the periodic orbits is that of the maximal and minimal itineraries, we can easily compute periodic orbits of arbitrary length for the system.

$$\begin{array}{rcl} 0 < M < x_l < L < & x_d & < R < x_h < N < 1 \\ \mathbf{\Phi}_l & \leq & \mathbf{L} \\ \mathbf{\Phi}_l \leq \mathbf{M} \leq & \Phi_0 & \leq \mathbf{N} \leq \Phi_h \\ \mathbf{R} & \leq & \Phi_h \end{array} \quad (4)$$

A further simplification could be included if we notice that the regions $M$ and $N$ should have the same properties, in the sense that they are essencially the same but separated by *chance* because of the discontinuity, so we could assign the same symbol for both. On our current discution we will continue using the 4 symbols though.

### C. Numerical representation of the Symbolic Words

It is custom to assign an integral numerical value to each of the symbols in the symbolic dynamics. In this way we may obtain a numerical equivalent to a sequence of symbols, which, making use of the property of the symbols being well arranged, allows us to make a good initial guess of the values of $x$ for our periodic points, once we have found the valid periodic sequences. The usual way to do this is to expand each number in its fraction expansion[11]which is very easy to implement. The problem with using such expansion is that the partition originating it has no relation with the partition we are using, even more, a value obtained with the use of this partition has no necesary meaning in the system we are working with, nevertheless, the ordering is preserved, which means that the numerical value of a sequence $MLR$ will be smaller than the value of a sequence $LMR$, but these numerical values will have nothing to do with an $x$ whose symbolic future is described by either sequence.

Let us take a look at the procedure to obtain a numeric equivalent of a word by using first the fraction expansion, and then let us contrast this with the procedure to obtain the number by means of the $\mathcal{N}$ and $\mathcal{W}$ functions, as will be discussed later.

*Fraction Expansion*

Consider a word $W = S_1 S_2 \dots S_n$, which is composed by $n$ symbols of an $m$-symbols alphabet. We may express the numerical equivalent of $W$ by means of a function $\mathcal{V}_m$ such that

$$\mathcal{V}_m(W) = \mathcal{V}_m(S_1 S_2 \dots S_n) = \sum_{i=1}^{n} \mathcal{N}_m(S_i) \frac{1}{m^i} \quad (5)$$

where $\mathcal{N}_m$ goes into $\{0, 1, \dots, m\}$, and $\mathcal{N}_m(S_i) = k - 1$ if $S_i$ corresponds to the $k$-th symbol of $\mathcal{A}$ and the number of order inverting symbols prior to $S_i$ is even; when the number of order inverting symbols prior to $S_i$ is odd, $\mathcal{N}_m(S_i) = m - k$.

This way we have found a numerical representation for the word $W$, which preserves the order and lies in the unit interval.

*Method of Different Weights*

In an attempt to be able to obtain a better initial guess of the location of our periodic points, we may attempt to try a different way of partitioning our unit interval. One good start would be to make use of the special points we use to partition the unit interval to build our symbolic dynamics. For instance, let us say that the intervals $[0, x_l]$, $[x_l, x_d]$, $[x_d, x_h]$ and $[x_h, 1]$ will correspond to the symbols $M$, $L$, $R$ and $N$, and that each of these intervals will have a weight given by the size of the interval. We may define two functions, $\mathcal{N}(S)$ and $\mathcal{W}(S)$, which go from our alphabet, $\mathcal{A}$ into a number in the unit interval as follows

$$\begin{aligned} \mathcal{N}(S_i) &= x_i \\ \mathcal{W}(S_i) &= \pm(x_{i+1} - x_i) \end{aligned} \quad (6)$$

where $x_i \in \{0, x_l, x_d, x_h, 1\}$, $S_i \in \{M, L, R, N\}$ and where $\mathcal{W}$ takes the negative sign whenever the symbol $S_i$ corresponds to a order inverting interval (ie. a region with negative slope). By means of these functions we may now attempt to expand any number in the unit interval, provided that at least one of the points, $x_i$, is greater than 0.5 and also one is smaller than 0.5.

We may now attempt to do a similar job than that done in the previous section, now using the defined in (6)

$$\begin{aligned} \mathcal{V}(S_i) &= \mathcal{N}(S_i) \\ \mathcal{V}(W) &= \mathcal{V}(S_1 S_2 \dots S_n) \\ &= \mathcal{N}(S_1) \\ &+ \begin{cases} \mathcal{W}(S_1)\mathcal{V}(S_2 S_3 \dots S_n), & \text{for } \mathcal{W}(S_i) > 0 \\ -\mathcal{W}(S_1)(1 - \mathcal{V}(S_2 S_3 \dots S_n)), & \text{for } \mathcal{W}(S_i) < 0 \end{cases} \end{aligned} \quad (7)$$

including an absolute value we may get rid of the conditional and have

$$\begin{aligned} \mathcal{V}(W) &= \mathcal{N}(S_1) + \left( \frac{|\mathcal{W}(S_1)| - \mathcal{W}(S_1)}{2} \right) \\ &+ \mathcal{W}(S_1)\mathcal{V}(S_2 S_3 \dots S_n) \\ &= \sum_{i=1}^{n-1} \left( \mathcal{N}(S_i) + \frac{|\mathcal{W}(S_i)| - \mathcal{W}(S_i)}{2} \right) \prod_{j=1}^{i-1} \mathcal{W}(S_j) \\ &+ \mathcal{N}(S_n) \prod_{j=1}^{n-1} \mathcal{W}(S_j) \end{aligned} \quad (8)$$

Let us analyse the form of the function $\mathcal{V}$ a little further. When dealing with variable size intervals, the weighing function $\mathcal{W}$ is telling us how big is the region of the unit interval corresponding to the symbol $S_i$, it also has information of the order of the symbolic sequences. The product of the $\mathcal{W}(S_j)$'s shrinks the unit interval according to the relative weight of the sequence of symbols already visited. The function $\mathcal{N}$ is telling us about the relative positions of the symbols within the shrunk interval.

The numerical value we can find by means of the function $\mathcal{V}$ also preserves the order, lies in the unit interval and even more, also allows us to represent any number within the unit interval by a word and viceversa.

### D. Periodic Orbits

In order to calculate averages of the system it is necesary to calculate the periodic orbits of the system. The periodic orbits are defined according to our symbolic dynamics as a sequence of symbols, describing the time evolution of the system, that repeats itself identically infinitelly many times. Take for instance the sequence $W = \overline{LNR}$, which, according to our method of different weights corresponds to the numerical value $\mathcal{V}(W) = 0.5864\ldots$ occurs in the system for $x_n = 0.59007\ldots$. Because of the way we defined the method of different weights we guarantee that the first symbol corresponding to the iteration of the numerical value, $\mathcal{V}(W)$, is the same than that of the original sequence, $L$ in our example. This may lead us to think that we should be able to use this initial guess for the $x$ for which the periodic orbit occurs in order to find the exact value of it. As explained in [5], we can use several ways to accomplish this, interval splitting method, Newton's method or multipoint shooting method for finding zeros of a function. In this project the Multipoint shooting method is followed because we have access to all the initial guesses at the same time, which means, we can calculate the numerical values of the words $\overline{LNR}$, $\overline{NRL}$ and $\overline{RLN}$ simultaneously.

It is important to notice that the methods to find the periodic orbits are usually highly sensible to the initial guesses we provide them, specially when the zeros of the functions we are trying to solve are too close together. This would mean that if the method of different weights does not give a value, $\mathcal{V}$, for the guess increasingly close to the real value, $x_n$, for which the periodic orbit occurs the initial guesses provided by it will not be good enough for our root finding algoritm.

The usual approach to this problem is to provide the root finding algorithm with some "stupid" guess for the shorter orbits, which can usually be guessed by hand, and then, use the calculated values to generate more accurate guesses for words of larger length. This method is foolproof but requires us to calculate all the periodic orbits up to certain period in order to calculate the next order of periodic orbits. On the other hand, the method proposed here lets us skip the requirement of finding all the short periodic cycles before attempting to calculate longer ones.

**Multipoint shooting method**

The multipoint shooting method is an improvement from the single point shooting method. It consist in guessing all the points compounding the periodic orbit at once and solving as many equations as the length of the periodic orbit simultaneously. This method requires more input than the others, but also converges faster and does not have too serious divergence problems since all the points are evolved only one step in time. To use the multipoint shooting method we need to solve the system of equations

$$\begin{aligned} f(x_1) &= x_2 \\ f(x_2) &= x_3 \\ &\cdots \\ f(x_{n-1}) &= x_n \\ f(x_n) &= x_1 \end{aligned} \tag{9}$$

We can define an $n$-dimensional vector function $F$

$$\vec{F}(\vec{x}) = \vec{F}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 - f(x_n) \\ x_2 - f(x_1) \\ \vdots \\ x_{n-1} - f(x_{n-2}) \\ x_n - f(x_{n-1} \end{pmatrix}, \tag{10}$$

and then calculate the Newton step as

$$\frac{d}{dx}\vec{F}(\vec{x})\left(\vec{x'} - \vec{x}\right) = -\vec{F}(\vec{x}), \tag{11}$$

where $\frac{d}{dx}\vec{F}(\vec{x})$ is given by

$$\frac{d}{dx}\vec{F}(\vec{x}) = \begin{pmatrix} 1 & & & & -f'(x_n) \\ -f'(x_1) & 1 & & & \\ & \cdots & 1 & & \\ & & \cdots & 1 & \\ & & & -f'(x_{n-1}) & 1 \end{pmatrix}. \tag{12}$$

The results obtained by applying this method are sumarized in the table

TABLE I: Points for period 2 through 4

| Word | $x$ | Word | $x$ |
|---|---|---|---|
| $\overline{MR}$ | 0.19996204857724 | $\overline{MLLR}$ | 0.44249154318464 |
| $\overline{LR}$ | 0.54387330877828 | $\overline{MLRR}$ | 0.28725599603236 |
| $\overline{MLR}$ | 0.30984744271745 | $\overline{LLLR}$ | 0.44249154318464 |
| $\overline{LLR}$ | 0.48008542459067 | $\overline{LLRR}$ | 0.49033618830224 |
| $\overline{LRR}$ | 0.56744169038733 | $\overline{LLNR}$ | 0.50525856910610 |
| $\overline{LNR}$ | 0.59007207981232 | | |

We can also see how the initial guesses compare with the actual values of $x$ for some of the shortest cycles.

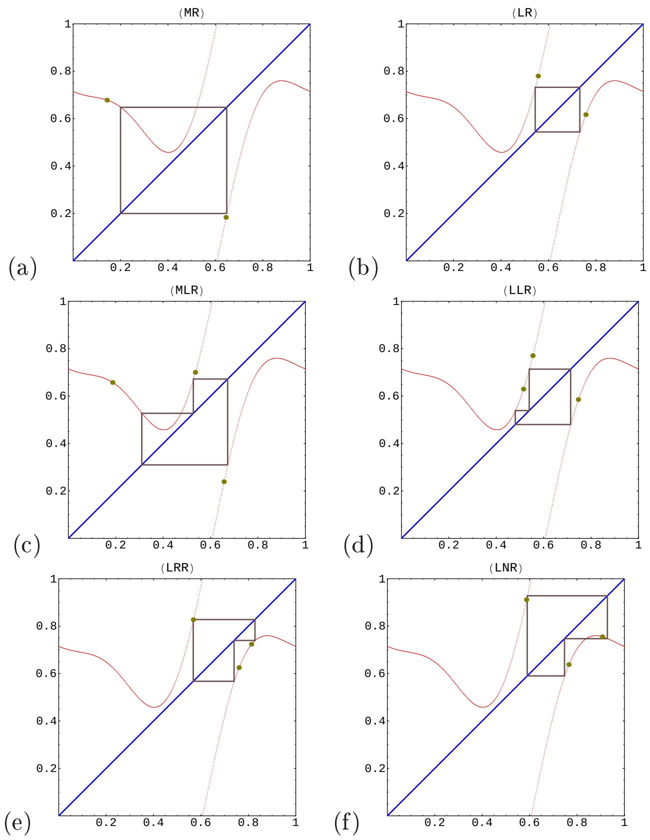


FIG. 6: Solid dots are initial guesses. Solid right lines are cycles.

## IV. DISCUSSION

The ultimate goal of the project was to calculate averages for the $1D$-circle map using the methods described in detail in [5], this meant to compute the periodic orbits of the system up to certain length, depending on the desired precision. To calculate the periodic orbits it was necesary to set up a suitable symbolic dynamics, study its topological properties (prunning rules, etc...) and write a computer program that would calculate with high accuracy the values for which the $1D$-map presents periodic orbits.

I studied the $1D$-map in detail and came up with an *original idea*[12] and developed a mathod that seemed promissing. I wrote a program that would allow me to calculate the periodic orbits for any circle-map, according to the method above mentioned, and that would make use of the pruning rules to generate all the valid word sequences for an specific given map.

It was realistic to expect to be able to calculate some averages by making use of the calculated periodic orbits and doing so would allow us to compare the results obtained by this method with the results obtained by studying the $2D$-Poincaré map of the original system.

During the course of this project I encountered a few problems, specially when attempting to calculate the periodic orbits with high accuracy. Eventhough the program that would calculate the periodic orbits seemed to be working as it should, the initial guesses provided to the multi-shooting method to find them needed an increasing accurary that my first order approximation to the points through the Method of Different Weights did not provide. The problem became evident when the program stopped working smoothly for longer orbits (period 14), but sadly, this was not the first moment where the problems had actually occurred. It can be seen on the table I that a problem occurs as soon as with the period 4 orbits, because the method used returns the same value for the calculation of the periodic orbit of two different words, $\overline{MLLR}$ and $\overline{LLLR}$. Early errors like this one make the method unreliable and force us to take different approaches when calculating the periodic orbits.

There is still work to do on this project, which includes calculating the averages and comparing them with those obtained for the driven pendulum. The question, *why are there convergence problems when calculating averages on the driven pendulum?* may have part of its answer buried in the analysis of simpler models as the one proposed here.

### Acknowledgments

I would like to thank Doctor Dr. Slaven Peleš for suggesting me this problem, his numerous advices helped me a lot going through the problem. I would also like to thank Doctor Predrag Cvitanovic, his love for the subject is contagious and his advice and what I learned through his lectures was unvaluable. At last I would like to thank my wife Lina Merchan, whose constant motivation and encouraging as well as intelligent suggestions make any project an easier goal.

## PROJECT PLAN

| Date | Progress |
|---|---|
| Thu Oct 23 2003 | A good $1D$-fit for the $2D$-map. |
| Thu Oct 30 2003 | Define the pruning and ordering rules. |
| Thu Nov 06 2003 | Write a program the allows the calculation of periodic orbits of length $N$ |
| Thu Nov 13 2003 | Begin calculating physical averages on the $1D$-map. |
| Thu Nov 20 2003 | Compare results obtained on the $1D$-map with those of the $2D$-map. |
| Thu Nov 27 2003 | Attempt to explain the reason for encountering difficulties calculating averages on the driven pendulum. |
| Thu Dec 04 2003 | Start writing the final report. |
| Thu Dec 11 2003 | Term paper deadline. |


[1] Randy Kobes, Junxian Liu and Slaven Peles, *Analysis of a Parametrically Driven Pendulum*, Physical Review E63, 03621 9, (2001).

[2] Zheng, Wei-Mou, *Kneading Plane of the Circle Map*, Chaos, Solitons & Fractals Vol. 4, No. 7, pp. 1221-1233 (1994).

[3] Gunaratne, Gemunu H., *Universality beyond the onset of chaos*, Preprint. pp. 37-69 (1989).

[4] Zheng, Wei-Mou, *Symbolic Dynamics for the Circle Map*, Int. J. Mod. Phys. B5 pp. 481-495 (1991).

[5] P. Cvitanović *et al.*, *Chaos: Classical and Quantum*, advanced graduate e-textbook, available online at `ChaosBook.org` (Niels Bohr Institute, Copenhagen 2003).

[6] From now on whenever we refeer to the region $M$, $L$, $R$ or $N$ we will mean all the points sharing the respective symbol

[7] An itinerary would be the sequence of points $\{f^{(n)}(x_0)\}$ that follows after iterating the map $n$ times starting on with the initial condition $x_0$

[8] A periodic orbit constitutes an infinitely long itinerary for it repeats infinitely many times.

[9] Eventhough a periodic orbit of length $n$ will have $n$ different values for the above called *exact value* for which it occurs, the symbolic sequence of each of these is just a shift of the symbols on it

[10] This rules are prescribed by the dynamics of the system.

[11] The expansion in $\frac{1}{2}$ of 0.75, $0.75 = 1\frac{1}{2} + 1\left(\frac{1}{2}\right)^2$.
The expansion of 0.5 in $\frac{1}{3}$, $0.5 = 2\frac{1}{3}+2\left(\frac{1}{3}\right)^2+2\left(\frac{1}{3}\right)^3+\cdots$.

[12] In the consulted literature I did not find any reference to this treatment to the problem.