Chapter 1

Outline of the boundary layer method

1.1 Motivation: Boundary layer separation and free surface flows

For a large variety of fluid flows, the creation of boundary layers near solid surfaces plays a major role. Even if flow in the bulk of the fluid is largely potential, the vorticity created in the boundary layer can penetrate into the bulk of the fluid by the phenomenon of boundary layer separation. In the process of separation, the boundary layer leaves the surface, and a recirculating flow appears. The motion of the fluid is changed dramatically if the separation of boundary layer occurs.

Another important problem is the description of a fluid having a free surface. The volume occupied by the fluid can change, and the shape of the free surface has to be found in connection with the motion of the bulk of the fluid, which makes the modelling of such flows even on a computer difficult. Therefore, it is very important to develop simple but accurate models for the flows with a free surface.

The goal of the following chapters (1-5) is to derive models applicable to free surface flows with boundary layer separation. There are several physical examples where such flows are realized. The circular hydraulic jump, appearing when a jet of fluid hits a surface and spreads out radially (an experiment which can be easily performed in the kitchen sink), exhibits separation of boundary layer at the rigid surface. The phenomenon of the circular hydraulic jump will be studied in detail in chapter 2. Wave-breaking can be viewed as another example of flows with a free surface exhibiting separation. One can perceive the process of wave breaking as a separation of the "surface boundary layer". The waves can also show separation of boundary layer at the bottom as the circular hydraulic jump does. The models for describing waves moving down an inclined plane will be developed in chapter 3.

We shall be mostly concerned with *stationary* flows, or non-stationary flows where the time dependence is simple (solitary waves). The keystone of our models will be the boundary layer approximation. Therefore, we shall devote this introductory chapter to the derivation of this approximation.

1.2 Governing equations

The motion of an incompressible viscous Newtonian fluid is described by the Navier-Stokes equations, the ith component of which reads

$$\frac{\partial}{\partial t}u_i + u_j\frac{\partial}{\partial x_j}u_i = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\Delta u_i, \tag{1.1}$$

where i ranges from 1 to the dimension of the space (usually, 2 or 3), Δ is the spatial Laplace operator, ν is the kinematic viscosity, u_i are the components of the velocity, ρ is the fluid density and p is the pressure. Summation over the index j is assumed.

Usually, equations (1.1) are brought into a dimensionless form by choosing a typical length scale L and velocity scale U. The spatial variables are then rescaled by L, velocity by U, time by L/U and pressure p by ρU^2. In this case, in the new dimensionless variables the Navier-Stokes equations (1.1) become:

$$\frac{\partial}{\partial t}u_i + u_j\frac{\partial}{\partial x_j}u_i = -\frac{\partial p}{\partial x_i} + \frac{1}{R}\Delta u_i. \tag{1.2}$$

Here

$$R = \frac{UL}{\nu} \tag{1.3}$$

is the Reynolds number.

The incompressibility condition is

$$\frac{\partial u_j}{\partial x_j} = 0. \tag{1.4}$$

Again, summation over the index j is assumed. The boundary conditions require that *all* the components of the velocity u_i vanish at the rigid surface S (no-slip conditions):

$$u_i(S) = 0. \tag{1.5}$$

In practical applications it is often important to consider the limit of very large Reynolds numbers $R \gg 1$. Let us, for example, estimate the value of R for a ship moving through water. The typical length scale is $L = 10$ meters, the typical velocity is $U = 1$ m/s (corresponding to a rather slow vessel), and the water viscosity is 10^{-6} m^2/s, thus the Reynolds number is $R \sim 10^7 \gg 1$.

One would naively think that when R is very large, the Laplace term in equation (1.2) can be simply neglected. If we just drop the Laplace term in (1.2), the highest derivatives will disappear and we get the Euler equations instead:

$$\frac{\partial}{\partial t}u_i + u_j\frac{\partial}{\partial x_j}u_i = -\frac{\partial p}{\partial x_i}. \tag{1.6}$$

In this approximation, it becomes impossible to fulfill the boundary conditions (1.5). To make the problem well-posed, one has to weaken the boundary condition (1.5) by requiring that only the component of the velocity *normal* to the surface vanishes:

$$u_n(S) = 0. \tag{1.7}$$

A simple example illustrates this important difference.

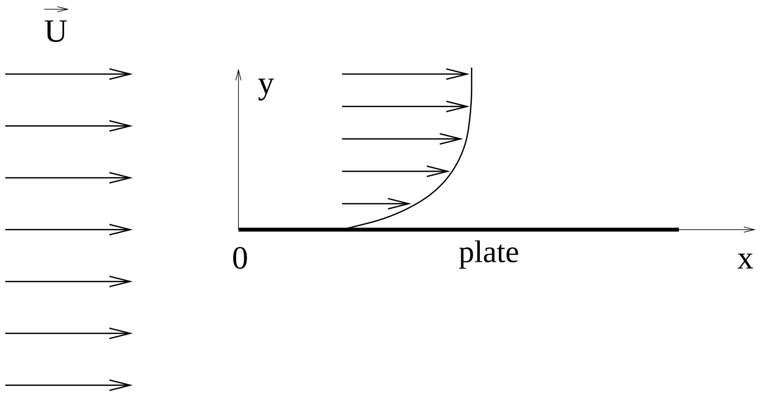

Figure 1.1: A half-plane inserted in a uniform stream (Blasius problem).

Consider a uniform two-dimensional flow of fluid having a constant velocity $U = (1, 0)$ everywhere. We put a half-plane into the fluid starting at $(0, 0)$ and lying along the x axis (see figure 1.1). An inviscid fluid flow satisfying (1.6) and (1.7) will not feel such a surface, because motion of the fluid is tangential to the surface everywhere. However, if a fluid is slightly viscous (however small the viscosity, or, equivalently, however large the Reynolds number R), both the tangential and normal velocities at the surface must vanish due to (1.5), and the velocity profile will no longer be uniform, as illustrated in figure 1.1. Thus, the overall structure of the flow is now changed. Can one introduce some sort of asymptotic theory describing solutions of (1.2) in the limit $R \to \infty$, which will be easier to deal with than (1.2)? This problem was solved by Prandtl [1, 2]. He assumed that the viscous forces are important only in a very thin layer close to the surface of the body (the boundary layer). The thickness δ of this boundary layer can vary along the surface, but is assumed to remain small. Outside the boundary layer (distances much larger than δ) the flow is inviscid and satisfies the Euler equation (1.6) with the boundary conditions (1.7).

For simplicity, we consider a two-dimensional flow around a body moving in a fluid. The coordinate system (x, y) is chosen in such a way that the line $y = 0$ propagates along the body of the surface, the lines $x = const$ are orthogonal to the surface at $y = 0$, and the system of coordinates (x, y) is orthogonal. This system of coordinates is similar to the one introduced in figure 1.1, but it has to be curvilinear if the surface of the body is curved. Let us call the x and y components of the velocity u and w, respectively. In this case, the system (1.2),(1.5) can be written explicitly as:

$$
\begin{aligned}
& u_t + uu_x + wu_y = -p_x + \frac{1}{R}(u_{xx} + u_{yy}) \\
& w_t + uw_x + ww_y = -p_y + \frac{1}{R}(w_{xx} + w_{yy}) \\
& u_x + w_y = 0.
\end{aligned}
\tag{1.8}
$$

Since the thickness of the boundary layer δ is assumed small, the variations of the flow in the y direction close to the surface are much faster then the variations of the flow in the x direction:

$$
\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x}. \tag{1.9}
$$

From (1.9) and the incompressibility condition (1.8.3) it follows that

$$w(r,z) \ll u(r,z). \tag{1.10}$$

To facilitate this difference in scales, we rescale the variables x and y according to

$$x = X,\ y = \delta Y. \tag{1.11}$$

The space derivatives then have to be transformed according to

$$\begin{aligned} \partial_x &= \partial_X \\ \partial_y &= \partial_Y/\delta. \end{aligned} \tag{1.12}$$

Also, according to the incompressibility equation (1.8.3), the velocities have to be rescaled as

$$u = U,\ w = \delta W. \tag{1.13}$$

The pressure is not scaled under the transformation (1.11), i.e., $p = P$. Substituting (1.13) and (1.12) into (1.2), we get

$$\begin{aligned} &U_t + UU_X + WU_Y = -P_X + R^{-1}(U_{XX} + \delta^{-2}U_{YY}) \\ &\delta(W_t + UW_X + WW_Y) = -\delta^{-1}P_Y + R^{-1}(\delta W_{XX} + \delta^{-1}W_{YY}) \\ &U_X + W_Y = 0. \end{aligned} \tag{1.14}$$

From equation (1.14.1), it is clear that the thickness of the boundary layer δ must be of the order $R^{-1/2}$. If we set $\delta = R^{-1/2}$, equations (1.14) become:

$$\begin{aligned} &U_t + UU_X + WU_Y = -P_X + R^{-1}U_{XX} + U_{YY} \\ &R^{-1}(W_t + UW_X + WW_Y) = -P_Y + R^{-2}W_{XX} + R^{-1}W_{YY} \\ &U_X + W_Y = 0. \end{aligned} \tag{1.15}$$

In the limit $R \to \infty$, we neglect the terms of the order R^{-1} and higher in (1.15) to obtain the *boundary-layer approximation* of (1.15):

$$\begin{aligned} U_t + UU_X + WU_Y &= -P_X + U_{YY} \\ 0 &= -P_Y \\ U_X + W_Y &= 0. \end{aligned} \tag{1.16}$$

The *internal* flow (the flow inside the boundary layer) described by (1.16) in the limit $Y \to \infty$ has to match the *external* flow given by (1.6) in the limit $y \to 0$. If $u_e(x,y)$, $p_e(x,y)$ is a solution of (1.6), we denote $U_e(X) = \lim_{y\to 0} u_e(x,y)$ and $P_e(X) = \lim_{y\to 0} p_e(x,y)$.

The external velocity $U_e(X)$ and pressure $P_e(X)$ are connected by means of the Bernoulli integral:

$$\frac{1}{2}U_e^2(X) + P_e(X) = const. \tag{1.17}$$

The matching condition for the internal solution $U(X,Y), P(X,Y)$ becomes:

$$\begin{aligned} &U(X, Y \to \infty) \to U_e(X) \\ &P(X, Y \to \infty) \to P_e(X), \end{aligned} \tag{1.18}$$

From (1.15.2) , we have $P_Y = 0$, or $P = P(X)$: the pressure is constant across the flow. From the matching conditions (1.18) for the pressure, $P(X,Y) = P_e(X)$: the pressure inside the boundary layer is given by the external pressure. Using (1.17), system (1.15) can be transformed into the following form:

$$\begin{aligned} &U_t + UU_X + WU_Y = U_e(X)U_e'(X) + U_{YY} \\ &U_X + W_Y = 0. \end{aligned} \tag{1.19}$$

The boundary conditions are the no-slip conditions at $Y = 0$ and the matching conditions at $Y = \infty$:

$$\begin{aligned} &U(Y = 0) = W(Y = 0) = 0 \\ &U(X, Y \to \infty) \to U_e(X). \end{aligned} \tag{1.20}$$

Equations (1.19) together with (1.20) describe Prandtl's model for the boundary layer flow.

1.2.1 A classical solution: Blasius profile

To illustrate the theory described above, we shall solve the problem posed in figure 1.1 by the boundary layer method. This problem of a flow past a flat plate is called the Blasius problem [3].

For a flat plate, as we have already noticed, the external inviscid flow is constant: $U_e(X) = 1$, thus the external pressure gradient is absent from (1.19). The flow is assumed to be stationary, thus $\partial_t = 0$ in (1.19). Equations (1.19) then simplify to

$$\begin{aligned} &UU_X + WU_Y = U_{YY} \\ &U_X + W_Y = 0 \end{aligned} \tag{1.21}$$

and the boundary conditions (1.20) are unchanged. We can introduce the streamfunction $\psi(X,Y)$ as

$$U = \psi_Y, \; W = -\psi_X. \tag{1.22}$$

The incompressibility condition (1.21.2) is satisfied identically. Equation (1.19.1) transforms to

$$\psi_Y \psi_{YX} - \psi_X \psi_{YY} = \psi_{YYY}, \tag{1.23}$$

and the boundary conditions for the streamfunction ψ are:

$$\begin{aligned} &\psi(Y=0) = \psi_Y(Y=0) = 0 \\ &\psi_Y(Y \to \infty) \to 1. \end{aligned} \tag{1.24}$$

We assume the following ansatz for the streamfunction

$$\psi = \sqrt{X} f(Y/\sqrt{X}). \tag{1.25}$$

Then, equation (1.23) becomes an *ordinary* differential equation for the function f:

$$-\frac{1}{2} f f'' = f'''. \tag{1.26}$$

Here the prime denotes the derivative of the function f with respect to its argument $\xi = Y/\sqrt{X}$. The boundary conditions for the functions $f(\xi)$ are

$$\begin{aligned} &f(0) = f'(0) = 0 \\ &f'(\xi \to \infty) \to 1. \end{aligned} \tag{1.27}$$

The boundary-value problem (1.26)-(1.27) can be solved numerically by the shooting and matching method. Adjusting $f''(0)$, one can achieve the desired asymptotic (1.27.2). The value of $f''(0)$ appears to be close to 0.032. The skin friction at the plate is thus $\tau(X) = U_Y(X, Y=0) \simeq 0.032 X^{-1/2} > 0$.

We can define the boundary layer thickness δ_* in the rescaled coordinates (X, Y) as

$$\delta_*(X) = \int_0^\infty \left(1 - \frac{U(X,Y)}{U_e(X)}\right) \mathrm{d}Y. \tag{1.28}$$

The thickness δ_* represents the displacement of the streamlines of the external flow due to the growth of the boundary layer and is therefore called *the displacement thickness*. Since $U(X,Y)$ approaches the asymptotic limit $U_e(X)$ very fast as $Y \to \infty$, the integral in (1.28) converges. Once the numerical solution of (1.26) is computed, the displacement thickness can be calculated from (1.28):

$$\delta_*(X) \simeq 1.729\sqrt{X}. \tag{1.29}$$

There are also other ways to define the boundary layer thickness. For example, one could define the boundary layer thickness (we shall call it h to distinguish it from (1.29)) in such a way that the velocity $U(X,Y)$ has 'almost' reached the asymptotic value $U_e(X)$ at $Y = h$. It is common to define the boundary layer thickness h so that the difference $U(X,Y)$ at $Y = h$ and $U_e(X)$ is 1 per cent:

$$U(X, Y = h) = 0.99 U_e(X). \tag{1.30}$$

Usually, the boundary layer thickness h defined by (1.30) is several times larger than the displacement thickness δ_* defined by (1.29).

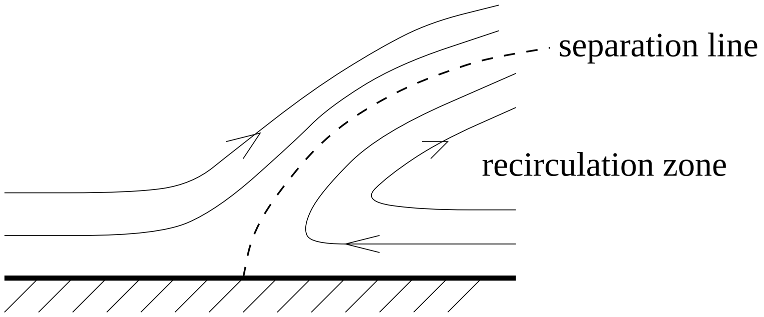

Figure 1.2: Separation of boundary layer.

1.2.2 Separation of boundary layer

It can happen that at some point along the boundary the flow becomes 'separated': the streamlines which were close to the boundary suddenly run away from it, and the reverse flow sets in (see figure 1.2). As was discovered by Goldstein [4], the boundary layer equations (1.19) develop a singularity at the point of separation. Physically, this is due to the fact that the boundary layer approximation (1.9-1.10) breaks down at the point of separation. Mathematically, this is manifested in the appearance of the singularity

$$U \sim (X_* - X)^{1/2} \text{ and } W \sim (X_* - X)^{-1/2}, \tag{1.31}$$

where X_* is the point of separation. To show this, we define the streamfunction $\psi(X, Y)$ according to (1.22) and expand this streamfunction in a power series in Y:

$$\psi = \sum_{k \geq 2} a_k(X) Y^k. \tag{1.32}$$

Then, the components of the velocity U and W are given by:

$$\begin{aligned} U &= \textstyle\sum_{k\geq 2} k a_k Y^{k-1} \\ W &= -\textstyle\sum_{k\geq 2} a_k' Y^k. \end{aligned} \tag{1.33}$$

The separation point X_* is defined as the point of vanishing skin friction:

$$\tau = \frac{\partial U}{\partial Y}|_{Y=0}(X = X_*) = 0. \tag{1.34}$$

From the first equation of (1.34), the separation occurs when the coefficient $a_2(X)$ in the expansion (1.32) vanishes. We remind the reader that for the Blasius solution described above the skin friction τ is equal to $\tau = 0.032/\sqrt{X} > 0$, thus the Blasius solution never separates.

Collecting terms of the same order in Y in (1.19) provides a system of equations for the coefficients a_k:

$$\begin{aligned}
Y^0 &: \quad a_3 = P_e'(X)/6 \\
Y^1 &: \quad a_4 = 0 \\
Y^2 &: \quad a_2 a_2' = 60 a_5 \\
Y^3 &: \quad \text{No terms of this order} \\
Y^4 &: \quad 3 a_3 a_3' = 210 a_7,
\end{aligned} \tag{1.35}$$

and so on. If $a_5 \neq 0$ at the point of separation $a_2 = 0$, then a_2 becomes singular: $a_2 \sim \sqrt{X_* - X}$, which corresponds precisely to the singularity of the type (1.31).

1.2.3 Marginal separation

There is a solution of the boundary layer equations (1.19) which does not become singular at the point of separation. If we assume that a_2' is finite at the point of separation, then from the second equation of (1.35), a_5 has to vanish simultaneously with a_2. In this case, a_6 also has to vanish and a_7 is prescribed by the Y^4 terms in (1.35). Continuing this procedure for increasing powers of Y, we conclude that a_8 and a_9 must be zero at the point of separation, and a_{10} and a_{11} are not zero. The factors a_k in the expansion (1.32) can be computed *ad infinitum*, once the pressure gradient $P_e'(X = X_*)$ and the derivative $a_2'(X = X_*)$ are given. Such a nonsingular solution of equations (1.19) is called *marginal separation*.

One can question the reliability of the marginal separation solution described above. As we have mentioned, the singularity in the boundary layer equations comes from the fact that the boundary layer approximation (1.9)-(1.10) breaks down at the point of separation. We thus believe that the solution describing the marginal separation is not physical, since a physically relevant solution of the boundary layer equations (1.19) *should* break down when the approximations of the theory become invalid.

A standard method nowadays for patching the boundary layer singularity is the triple deck theory [5]. These methods explicitly use the fact that it is possible to match the solution in the vicinity of the separation point asymptotically to the external flow. For separation in flows with a free surface in which we are interested, such a matching procedure is not possible. First, for flows with a free surface, the matching conditions (1.20) at $Y = \infty$ have to be substituted by the conditions on the free surface (to be derived later). Second, for the circular hydraulic jump, the separation eddy nearly reaches the free surface. We thus turn to other methods for dealing with the boundary layer singularity.

1.2.4 Approximate integral method by von Kármán and Pohlhausen

Initially, the integral methods were developed by von Kármán and Pohlhausen for computations of approximate solutions to the boundary layer equations (1.19) up to the separation point. However, it appeared that these methods were capable of describing a solution *immediately beyond* the separation point. We shall therefore first give an outline of the standard integral method of von Kármán and Pohlhausen and then mention why this method needs a modification to describe the separated velocity profile accurately.

The integral methods attempt to approximate the solution of (1.19) by a given profile, rather then trying to solve the equations exactly. One can, for example, try to approximate a solution of (1.19) $U(X,Y)$ by the following polynomial ansatz:

$$U = U_e(a\eta + b\eta^2 + c\eta^3 + d\eta^4), \tag{1.36}$$

where $\eta = Y/h$, and h is the boundary layer thickness defined by (1.30). U_e, as usual, denotes the velocity of the external flow.

Instead of imposing the boundary conditions at $Y = \infty$ according to (1.20) we consider boundary conditions at $Y = h$, i.e., at $\eta = 1$:

$$\begin{aligned} &U = U_e \\ &U_\eta = U_{\eta\eta} = 0. \end{aligned} \tag{1.37}$$

Of course, (1.37) is an approximation of the asymptotic matching condition in (1.20). However, since the solution $U(X,Y)$ at $Y = h$ has almost converged to $U_e(X)$ (we remind the reader that the definition of the boundary layer thickness h (1.30) requires that $U(X, Y = h) = 0.99U_e(X)$), (1.37) is valid with sufficiently high accuracy.

We impose the condition that the boundary layer momentum equation (1.19.1) has to be valid at $Y = 0$, which gives

$$U_e \frac{dU_e}{dX} = \frac{\partial^2 U}{\partial Y^2}\Big|_{Y=0}\ , \tag{1.38}$$

which can be further transformed into

$$U_e \frac{dU_e}{dX} = \frac{2U_e}{h^2} b. \tag{1.39}$$

Following [6], we define the *shape parameter* $\Lambda(X)$ as

$$\Lambda = h^2 \frac{dU_e}{dX}. \tag{1.40}$$

The parameter Λ is a prescribed function of X. The boundary conditions (1.37) together with (1.40) provide a linear system of equations for the coefficients a, b, c, d:

$$\begin{aligned} &a + b + c + d = 1 \\ &a + 2b + 3c + 4d = 0 \\ &2b + 6c + 12d = 0 \\ &b = -\Lambda/2. \end{aligned} \tag{1.41}$$

From (1.41), the coefficients a, b, c and d can be expressed in terms of Λ as

$$\begin{aligned} a &= 2 + \Lambda/6 \\ b &= -\Lambda/2 \\ c &= -2 + \Lambda/2 \\ d &= 1 - \Lambda/6. \end{aligned} \tag{1.42}$$

The only unknown is the thickness of the boundary layer $h(X)$. The equation for h can be obtained by integrating the momentum equation (1.19) from 0 to h and multiplying by h. We obtain the following expression:

$$h\frac{d}{dx}\left\{hU_e^2\int_0^1\left(\frac{U^2}{U_e^2}\right)\mathrm{d}\eta\right\} - hU_e\frac{d}{dx}\left\{hU_e\int_0^1\left(\frac{U}{U_e}\right)\mathrm{d}\eta\right\} = -hP_e' + U_\eta\Big|_0^1, \tag{1.43}$$

which can be rewritten in terms of Λ using (1.36), (1.40) and (1.42) as

$$h\frac{d}{dx}\left\{hF_2(\Lambda)\right\} - hU_e\frac{d}{dx}\left\{hU_e\left(\frac{3}{10}+\frac{\Lambda}{120}\right)\right\} = h\Lambda U_e - U_e(2+\frac{\Lambda}{6}). \tag{1.44}$$

Here

$$F_2(\Lambda) = \int_0^1\left(\frac{U^2}{U_e^2}\right)\mathrm{d}\eta = \frac{367}{630} + \frac{71\Lambda}{7650} + \frac{\Lambda^2}{9072}. \tag{1.45}$$

To solve (1.44), one needs to prescribe the velocity of the external flow $U_e(X)$, which will give $\Lambda(X)$ according to (1.40).

For example, for the Blasius problem (see figure 1.1), there is no external pressure gradient and the external velocity is constant $U_e = 1$, thus $\Lambda = 0$. In this case, (1.44) simplifies to

$$\frac{37}{315}hh' = 2. \tag{1.46}$$

The boundary layer thickness h thus grows as

$$h = \sqrt{\frac{1260X}{37}}. \tag{1.47}$$

Using (1.28) and (1.42), one can derive the explicit relationship between h and the displacement thickness δ_*:

$$\delta_* = \int_0^\infty\left(1 - \frac{U(X,Y)}{U_e(X)}\right)\mathrm{d}Y \simeq h\int_0^1\left(1 - \frac{U(X,\eta)}{U_e(X)}\right)\mathrm{d}\eta = h\left(\frac{3}{10} - \frac{\Lambda}{120}\right). \tag{1.48}$$

For the Blasius problem $\Lambda = 0$, and δ_* is given by

$$\delta_* = \frac{3}{10}\sqrt{\frac{1260}{37}X} \simeq 1.751\sqrt{X}. \tag{1.49}$$

There is less then a 1 per cent difference between (1.49), afforded by the integral method, and (1.29), obtained from the exact Blasius solution.

However, the most important feature of (1.44) is that it can describe a profile with a separation. In fact, if $\Lambda < -12$, the separation occurs, since the coefficient a in (1.42) becomes negative. One can see that even if $\Lambda < -12$, equation (1.44) does not become singular [6]. This can be explained by the following simple argument. The boundary layer approximation breaks down only in the immediate neighbourhood of the separation point. Across the flow sufficiently far from the separation point, the boundary layer approximation is still valid. Thus, if one considers averages of the momentum equation (1.19.1) across the flow, the singularity at the separation can be avoided.

Even though the standard integral method by von Kármán and Pohlhausen is capable of describing separated flows, the description *beyond* the separation point becomes inaccurate and the solution frequently diverges [6]. Physically, this is due to the fact that when the separation of boundary layer occurs, the 'effective shape' of the body changes. The external stream has to flow around the separation region, as is illustrated in figure 1.2. This changes the external pressure P_e and external velocity U_e. Thus, the boundary layer flow is coupled to the external flow after the point of separation, and this coupling is neglected in standard boundary layer equations (1.19) and, correspondingly, in the integral method of von Kármán and Pohlhausen. In the following chapters, we shall introduce this missing feedback between the boundary layer flow and the pressure. We shall mainly develop models for free surface flows. We shall demonstrate that flows exhibiting separation of boundary layer can be accurately described by such modified integral methods.

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