Chapter 6

Introduction to "Global bounds ..."

6.1 Preface

Spontaneous appearance of singularities in physical systems has always been a topic of active research in modern physics. It is not at all true that a well-posed physical problem should behave nicely if all the relevant physical effects are taken into account. One can think of problems where the solution *cannot* be smooth due to *physical* reasons. Consider, for example, the break-off of a drop of fluid from a faucet (this problem was studied in [1]). Before the break-off, the drop and the fluid constitute one piece, but immediately after the break-off, the bulk of the fluid inside the faucet and the drop are separated by a free surface. The *physics* of break-off is singular, and there is no reason to expect a smooth solution, no matter how accurate the model used to describe this problem is.

We should also mention that on occasion, a physical model exhibiting singularities clarifies the physical behaviour of the system and makes the solution of the problem algebraically easier. This can be illustrated by shock waves in gas dynamics. Of course, if one zooms down to a molecular scale, where the dissipation occurs, the solution is smooth, but for all practical purposes it is easier to consider the solution as a shock, since the dissipation region is very thin. The analytic treatment of a solution with a shock is greatly simplified, and a series of conservation laws across the shock can be derived.

In this chapter, we shall consider the formation of singularities for solutions of equations of the type

$$u_t = F(u, u_x, u_{xx}, \ldots; x, t). \tag{6.1}$$

The right-hand side of (6.1) depends on the spatial derivatives of the solution u. It can also depend on the spatial variable x and time t. First, we shall try to explain how to prove global bounds for the solution of equation (6.1), thus excluding certain types of singularities. Once this is done, we shall develop a theory for analysing the singularities in detail. After the general theory has been studied, we discuss the results obtained in paper [2], giving a general outline without going into too many details.

6.2 Integral inequalities and global bounds

If an equation of the type (6.1) is given, it is most natural to compute the evolution of integral quantities of the type

$$\frac{d}{dt}\int_{x-\text{domain}} |\partial_x^k u|^m \mathrm{d}x = (\ldots). \tag{6.2}$$

Here $m \geq 0$ and $k \geq 0$ are natural numbers. The absolute value sign is not necessary when m is an even number. The right-hand side of (6.2) is an integral of the function F from the right-hand side of (6.1), multiplied by some powers of the x-derivatives of u, and the integral is taken over the whole x region. Suppose, for example, that we managed to prove that the right-hand side of (6.2) is non-positive for some k and m:

$$\frac{d}{dt}\int_{x-\text{domain}} |\partial_x^k u|^m \mathrm{d}x \leq 0. \tag{6.3}$$

Then, the quantity

$$\int_{x-\text{domain}} |\partial_x^k u|^m \mathrm{d}x \tag{6.4}$$

is bounded by a constant (depending on the initial conditions), which forms a global bound for the solution. This global bound puts constraints on which types of singularity are admissible. If a singularity appears, it must be sufficiently weak so that the quantity (6.4) does not diverge. For example, if a singularity is of the kind $u \simeq (x-x_0)^p$, then the boundedness of (6.4) requires that

$$\int_{x \text{ close to } x_0} (x-x_0)^{m(p-k)} \mathrm{d}x \tag{6.5}$$

is finite, giving

$$m(p-k) > -1. \tag{6.6}$$

How does one get estimates of the kind (6.3)? What are the values of k and m to be chosen? There is no general way, and one has to try different possibilities. For $k = 0$, we multiply the left-hand side of (6.1) by u^{m-1} and integrate by parts. The boundary terms vanish if periodic boundary conditions are used. To get the estimate (6.2) with $k \geq 0$, one has to differentiate equation (6.1) k times with respect to x, multiply by $(\partial_x^k u)^{m-1}$ and integrate with respect to x. On the right-hand side, integration by parts has to be performed (see immediately below for an example). Various integral inequalities can be used to derive the estimate (6.3), see Appendix of [2].

We shall illustrate these ideas on a simple example of the heat-conductance equation:

$$u_t = u_{xx}. \tag{6.7}$$

Multiply both sides by u and integrate with respect to x over the whole x-domain, assuming periodic boundary conditions. We get

$$\frac{1}{2}\frac{d}{dt}\int u^2 \mathrm{d}x = \int u u_{xx} \mathrm{d}x = -\int u_x^2 \mathrm{d}x \leq 0. \tag{6.8}$$

The last equality follows by an integration by parts. The boundary terms vanish due to periodic boundary conditions. The expression (6.8) is precisely of the type (6.3) with $k = 0, m = 2$.

Equation (6.7) is sufficiently simple for the estimates of the type (6.3) to be derived for any natural k with $m = 2$. In fact, if we differentiate equation (6.7) k times, multiply by $\partial_x^k u$ and integrate with respect to x , we get

$$\frac{1}{2}\frac{d}{dt}\int(\partial_x^k u)^2 \mathrm{d}x = \nu\int(\partial_x^k u)(\partial_x^{k+2}u)\mathrm{d}x = -\nu\int(\partial_x^{k+1}u)^2\mathrm{d}x \leq 0. \tag{6.9}$$

An estimate even stronger than (6.3) can be derived, using estimates for the integral quantities. We shall illustrate this for (6.9). Due to the Poincaré inequality (see Appendix in [2]), we have

$$\int f^2\mathrm{d}x \leq L^{-2}\int f_x^2\mathrm{d}x, \tag{6.10}$$

for any function $f(x)$ such that $\int f(x)\mathrm{d}x = 0$. Thus, (6.9) can be transformed into

$$\frac{1}{2}\frac{d}{dt}\int(\partial_x^k u)^2 \mathrm{d}x \leq -\nu L^2\int(\partial_x^k u)^2\mathrm{d}x \leq 0. \tag{6.11}$$

We can conclude that the relevant integral quantity (6.2) decays at least exponentially:

$$\int(\partial_x^k u)^2\mathrm{d}x \leq Ce^{-\nu L^{-2}t}. \tag{6.12}$$

When applying this method, one has to take care of one very important detail. In the derivation of the estimates of the type (6.2), one usually integrates by parts to estimate terms on the right-hand side, as we did in (6.9). This, however, requires that the functions under the integral are sufficiently smooth. For our simple example (6.7), one can prove the smoothness (actually, infinite smoothness for $t > 0$) of solutions using the exact solution [5], so the estimates (6.9) can be written for any $k > 0$. However, proving the smoothness of solutions for a general non-linear equation (6.1) is a non-trivial task. In our paper [2], we have just assumed the sufficient smoothness (Assumption 1 in [2]) and used this assumption in all the Theorems and Lemmas proved in [2].

6.3 Complete and incomplete similarity solutions

The most interesting part of the paper [2] is, in our opinion, contained in section 4, dealing with the study of the singularity. We shall try to explain this section in more detail. We shall take a broader view on this problem and introduce the reader to the notion of *complete and incomplete similarities.* The similarity solutions are frequently used not only in studying the singularities of differential equations, but also in the theory of turbulence, crack propagation, field theory (renormalization group) and other areas (for an excellent review on the subject, see [4]).

6.3.1 Complete similarity solutions

The idea is to look for *particular* solutions of (6.1) of the type

$$\begin{aligned} u(x,t) &= (t_0 - t)^{\alpha} f((t_0 - t)^{\beta}x) \quad \text{or} \\ u(x,t) &= t^{\alpha} f(t^{\beta}x). \end{aligned} \tag{6.13}$$

Here $f(\xi)$ is a function with the argument ξ equal to

$$\begin{aligned} &\xi = (t_0 - t)^\beta x \quad \text{or, correspondingly} \\ &\xi = t^\beta x. \end{aligned} \tag{6.14}$$

Determining the function $f(\xi)$ is the main point of the method. To find $f(\xi)$, we substitute into (6.1). If the terms explicitly dependent on t can be cancelled and the equation written only in terms of $f(\xi)$, we say that equation (6.1) admits a self-similar solution. The requirement of the cancellation of terms explicitly depending on t usually provides an explicit relationship between the constants α and β in (6.13).

We illustrate these ideas for the simple example we have used before (6.7), assuming the boundary conditions

$$u(x = \pm\infty) = 0. \tag{6.15}$$

The exact solution of (6.7) is known [5]

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int u_0(x') e^{-\frac{(x-x')^2}{4t}} \mathrm{d}x', \tag{6.16}$$

where $u_0(x)$ is the initial condition $u(x, t = 0) = u_0(x)$. We shall investigate the solutions of (6.7) in the limit $t \to 0$ as if the expression (6.16) were not known, and then compare our conclusions with the exact result (6.16).

Which one of the formulas (6.13) should we take? Equation (6.7) describes the evolution of heat transported inside a one-dimensional body. Suppose that the initial heat energy is concentrated at the center, around $x = 0$. Since the ends are cold ($u(x = \pm\infty) = 0$), the temperature at the center will slowly decay and approach 0 everywhere in the limit $t \to \infty$, due to the diffusion of heat to $x = \pm\infty$. The expression (6.13.1) is defined only when $t < t_0$, thus we need to take (6.13.2) since the temperature distribution exists for all $t > 0$. The initial peak of temperature in the center will become more and more diffused with time, thus $\beta \leq 0$ and $\alpha \leq 0$ in (6.13.2). Assuming ansatz (6.13.2) for the solution and substituting it into (6.7), we obtain:

$$t^{\alpha-1}(\alpha f + \beta\xi f'(\xi)) = t^{\alpha+2\beta} f''(\xi). \tag{6.17}$$

To cancel the terms containing t on both sides, we need

$$\beta = -1/2, \text{ and } \alpha \text{ arbitrary .} \tag{6.18}$$

The self-similar function $f(\xi)$ satisfies the *ordinary* differential equation

$$\alpha f - \frac{1}{2}\xi f'(\xi)) = f''(\xi), \tag{6.19}$$

where we have substituted $\beta = -1/2$ from (6.18) into (6.17). The general solution of (6.19) can be expressed in terms of the parabolic cylinder functions:

$$f(\xi) = e^{-\xi^2/8} \left(C_1 D_{-2\alpha}(\sqrt{2}\xi) + C_2 D_{-2\alpha}(-\sqrt{2}\xi) \right), \tag{6.20}$$

where $D_p(x)$ is the parabolic cylinder function of order p, see [5], and the C's are arbitrary coefficients. For a general equation of the type (6.1) such an analytic solution is not usually available, and one has to solve the analogue of (6.19) numerically.

The choice of α in our example (6.7) is governed by the initial condition $u(x,t=0=u_0(x)$. For example, for a point source $u_0(x)=\delta(x)$ we have $\alpha=-1/2$. These results (along with all the initial conditions leading to a self-similar solution) are derived in the Appendix. There is no restriction on α, and ansatz (6.13.2) with $\beta=-1/2$ and any value of α can be realized. For a more general equation of the type (6.1), α may be chosen for different reasons (as in the case of asymptotic matching in the following section).

We observe that a very special initial condition is needed for the solution to be described by the ansatz (6.13.2) on the whole interval $-\infty<x<+\infty$. Thus, it looks as if the self-similar solution is very special and therefore of hardly any practical importance. This is, however, not true, since self-similar solutions can also describe the approach to a singularity from a *general* initial condition. The corresponding method (called *incomplete similarity*) is described in the following section.

6.3.2 Incomplete similarity

Again, we shall illustrate our ideas with the simple example of the heat-conducting equation (6.7). Suppose we start with the initial condition $u_0=\delta(x)$ and let the temperature $u(x,t)$ decay according to (6.16). We call this solution $\mathcal{E}(x,t)$ (which is the fundamental solution of the heat-conduction equation):

$$\mathcal{E}(x,t)=\frac{1}{2\sqrt{\pi t}}e^{-\dfrac{x^2}{4t}}. \tag{6.21}$$

At $t=1$, we suddenly set another point source in the middle: $u(x,t=1)=\mathcal{E}(x,t=1)+\delta(x)$. Since the heat-conduction equation is linear, the solution $u(x,t)$ for $t>1$ satisfies

$$u(x,t)=\mathcal{E}(x,t)+\mathcal{E}(x,t-1). \tag{6.22}$$

We make the time shift $\tau=t-1$. In this case, (6.22) becomes

$$u(x,\tau)=\mathcal{E}(x,\tau+1)+\mathcal{E}(x,\tau). \tag{6.23}$$

We shall study (6.23) in the limit when τ is small and positive. The graph of (6.23) for $\tau=0.01$ is shown in figure 6.1. When τ is small and positive, the regular part $\mathcal{E}(x,1+\tau)$ changes smoothly, and the singular part $\mathcal{E}(x,\tau)$ changes very rapidly as a function of x close to $x=0$. We shall 'zoom in' on the point $x=0$ using the following transformation

$$\begin{aligned}\xi&=x\tau^{-1/2}\\ \tau&=t-1.\end{aligned} \tag{6.24}$$

In the new variables ξ,τ the solution (6.23) can be rewritten as

$$u(\xi,\tau)=\frac{1}{2\sqrt{\pi(\tau+1)}}e^{-\dfrac{\xi^2\tau}{4(\tau+1)}}+\frac{1}{2\sqrt{\pi\tau}}e^{-\dfrac{\xi^2}{4}}. \tag{6.25}$$

Far away from $x=0$, the 'bump' appearing from the second term of (6.25) decays very fast as a function of x and only the first term contributes to $u(\xi,\tau)$ in (6.25). On the other hand, close to $x=0$ the second term of (6.25) changes very rapidly as a function

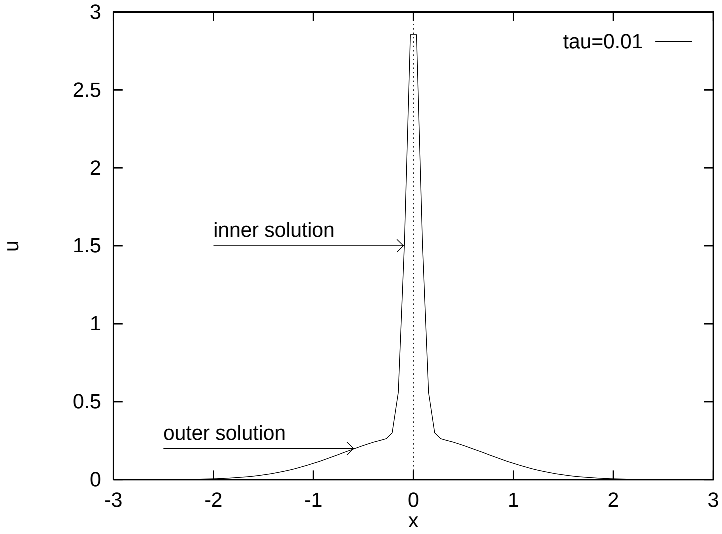

Figure 6.1: A plot of the solution (6.23) for $\tau = 0.01$ The inner and outer solutions are shown.

of x, while the first term remains almost constant. The size of this region of rapid change is of the order of $\tau^{1/2}$. Thus, the exact solution (6.25) can be divided into two parts: the 'inner' and the 'outer' part. The 'outer' solution is given by the first term in (6.25):

$$u_{\text{outer}}(x,\tau) = \frac{1}{2\sqrt{\pi(\tau+1)}} e^{-\dfrac{\xi^2\tau}{4(\tau+1)}} = \frac{1}{2\sqrt{\pi(\tau+1)}} e^{-\dfrac{x^2}{4(\tau+1)}}. \tag{6.26}$$

The 'inner' solution describes the rapid change in the region of the intermediate ξ, with the second term of (6.25) being approximated by a constant:

$$u_{\text{inner}}(\xi,\tau) \simeq \frac{1}{2\sqrt{\pi}} + \frac{1}{2\sqrt{\pi\tau}} e^{-\dfrac{\xi^2}{4}}. \tag{6.27}$$

The 'inner' solution (6.27) and the 'outer' solution (6.26) have to overlap in the region of intermediate ξ. Thus, we require that the limit of $\xi \to \infty$ of the inner solution (6.27) coincides with the limit $x \to 0$ of the 'outer' solution (6.26) for all τ:

$$\lim_{\xi\to\infty} u_{\text{inner}}(\xi,\tau) = \lim_{x\to 0} u_{\text{outer}}(x,\tau). \tag{6.28}$$

If the exact solution to the equation is not known (which is usually the case), one can still rely on the multiple-scale analysis described above to predict the form of the singularity. Ansatz (6.13) is assumed *only for the inner solution.* In this case, only the region close to the singularity is described by the self-similar ansatz (6.13) , whereas the solution elsewhere remains smooth. Thus, the following steps are needed to compute the incomplete similarity solution:

1. Identify the 'outer' solution $u_{\text{outer}}(x,t)$ and compute its limit when $x \to 0$.

2. Assume ansatz (6.13) for the solution close to the singularity, and solve for the function $f(\xi)$. The equation for the self-similar function $f(\xi)$ is obtained by substituting ansatz (6.13) into (6.1) and collecting the most singular terms when the solution approaches the singularity.

3. Find the dependence of the asymptotics of the function $f(\xi)$ on α and β. The condition for asymptotic matching (6.28) can sometimes select the values α and β in (6.13) (as is the case in our paper [2]).

We shall now illustrate these ideas for our simple example of the heat-conduction equation. The 'outer' solution is given by (6.26), which tends to $1/(2\sqrt{\pi})$ when $x \to 0$.

We seek the 'inner' solution in the form

$$u(x,\tau) = \tau^{\alpha} f(\tau^{\beta} x) + C_1, \tag{6.29}$$

where C_1, α and β are constants to be determined. Substituting (6.29) into the heat-conduction equation gives exactly (6.19), and due to the initial conditions of a point source we again get $\alpha = -1/2$. Thus, the solution for $f(\xi)$ is:

$$f(\xi) = Ce^{-\xi^2/4} + C_1. \tag{6.30}$$

The 'inner' solution tends to C_1 when $\xi \to \infty$, and the condition (6.28) gives $C_1 = 1/(2\sqrt{\pi})$.

The prefactor C in (6.30) can be determined in this case due to the total conservation of heat:

$$\int_{-\infty}^{+\infty} u(x,t)\mathrm{d}x = const. \tag{6.31}$$

Substituting $u = u_{\text{inner}}(\xi,\tau) + u_{\text{outer}}(\xi,\tau)$ into (6.31), we get $C = 1/(2\sqrt{\pi})$. We shall emphasise, however, that only the *global* conservation law can be used. It is incorrect to use the conservation law only for the 'inner' problem to determine the prefactor C in (6.30), since the heat can flow in and out of the singularity (although formally for this particular case we get the same answer). Generally, even if a conservation law for the system is known, there is no need for it to be satisfied in the 'inner' region. There can be exchange with the 'outer' region, so that the quantity in hand is conserved only *globally*.

This concludes our introduction to the theory of self-similarity and incomplete similarity. Now we are ready to discuss the results obtained in our article [2].

6.4 Statement of the problem

The equation studied in [2] is

$$u_t = (u_x^2)_{xx} - \nu u_{xxxx}. \tag{6.32}$$

Here t is time, x is the space coordinate and the subscripts denote the partial derivatives with respect to x and t. $\nu \geq 0$ is a parameter (called the hyperviscosity). We shall also use the differentiated version of (6.32) with $w = u_x$

$$w_t = (w^2)_{xxx} - \nu w_{xxxx}. \tag{6.33}$$

The finite-time singularities appear when $\nu = 0$ in (6.33), i.e., in the following equation

$$w_t = (w^2)_{xxx}. \tag{6.34}$$

Our study was motivated by [6], where the occurrence of finite-time singularities in (6.32) was reported for sufficiently small values of ν. We shall thus be concerned with the question to what extent the solutions of (6.32), (6.33) and (6.34) remain regular ("smooth") and what is the form of the emerging finite-time singularity, if the latter occurs.

6.5 Rigorous results obtained in "Global bounds..."

We believe the results in the paper are explained sufficiently, and no further introduction is needed for the reader to read through chapters 2 and 3. Thus, we shall now summarise the results obtained in the paper [2].

6.5.1 Rigorous results: $\nu = 0$

Section 2 of the article [2] is devoted to rigorous results in the inviscid case: $\nu = 0$ in (6.32) and (6.33). Mainly, this chapter is devoted to the proof of the conservation of

$$\int |u_x|^3 \mathrm{d}x = const, \tag{6.35}$$

which is done in Theorem 1. Expression (6.35) is of the type (6.3). Several steps are needed to prove (6.35), and these steps are summarised in Lemmas 1 and 2. First, we prove conservation of

$$\int_{b(t)}^{b(t)} u_x^3 \mathrm{d}x = const, \tag{6.36}$$

where the points $a(t)$ and $b(t)$ are two subsequent zeroes of u_x: $u_x(a(t),t) = u_x(b(t),t) = 0$. Differentiating (6.36) with respect to time gives 0 provided that $a'(t)$ and $b'(t)$ are continuous. Thus, proving the continuity of a' and b' is the next step. Next, we have to prove that the total number of zeroes of u_x is constant. Finally, the whole x-region is divided into subintervals $[a_i(t), a_{i+1}(t)]$, whose ends a_i are the subsequent zeroes of u_x: $u(a_i(t),t) = 0$. On these intervals, the integrals of the type (6.36) can be multiplied by the sign of u_x (inside every such interval, the $\mathrm{sign}(u)$ is just a constant equal to ± 1). Since the number of intervals (equal to the number of zeroes of u_x) is constant, the integral (6.35) can be obtained as a sum of all the pieces (6.36). After this is done, the proof of Theorem 1 and the corresponding global bounds for the solution becomes straightforward. The global bounds follow by the direct application of Sobolev inequalities and give an estimate for the absolute value of the solution. By analogy with (6.6), we obtain an estimate for which types of singularities are possible.

Also in section 2, we argue that the maxima of the solution of (6.32) sharpen but the valleys flatten out. Thus, if we start with a symmetric initial condition $u(x, t = 0) = u(-x, t = 0)$ with only one maximum at $x = 0$ (like a gaussian peak), the singularity can be expected only at $x = 0$. Knowing the exact position of the singularity will simplify our analysis in section 4 of the article [2].

6.5.2 Rigorous results: $\nu > 0$

Section 3 of [2] deals with rigorous results when the hyperviscous term is present $\nu > 0$ in (6.32-6.33). These results are summarised in Theorem 2. Parts 1 and 2 of this theorem can be derived in absolutely the same way as (6.8) and (6.12). Part 3 (proving (6.3) for $k = 1$ and $m = 2$) is substantially more sophisticated than the previous two, and the final result of Part 3 holds only if the initial conditions for equation (6.33) are very small. We shall not go into details about how Part 3 was proved, since the proof is quite formal and does not add much to our physical intuition. The reader can consult [2] for further details.

6.6 Analysis of the singularity

In this section, the general theory of obtaining incomplete self-similar solutions is applied to equation (6.34). First, the outer solution is computed (which appears to be a shock). Then, an inner solution of the type (6.13.1) is assumed and the self-similar function f is computed. Finally, the asymptotic matching to the outer solution is performed, selecting the values of α and β in (6.13.1)

6.6.1 Outer solution: shocks

Earlier, we established that if the initial condition of (6.32) with $\nu = 0$ has only one maximum and is symmetric around $x = 0$, the singularity should be expected at $x = 0$. This correspond to a zero of the solution of (6.34), when w changes from $+$ to $-$, moving from left to right. We have shown in [2] that the singularities of (6.34) are shocks, and w has to change sign. In fact, even if w is discontinuous, w^2 becomes continuous, and can even be made twice differentiable with respect to x by adjusting the higher derivatives of w at the shock.

The decay of a solution after the shock is formed is well-predicted by the separated variables ansatz. We assume

$$w(x,t) = T(t)F(x). \tag{6.37}$$

Substitution of this ansatz into (6.34) gives

$$\frac{T'(t)}{T(t)^2} = \frac{(F(x)^2)'''}{F(x)} = \lambda, \tag{6.38}$$

where λ is an unknown eigenvalue. The condition that $F(x)$ has a shock at $x = 0$ gives $F(0) = 1$, $F'(0) = 0$. The solution, assumed periodic on the interval $[-L/2, L/2]$, has to be antisymmetric around the point $x = L/2$, thus $F(L/2) = F''(L/2) = 0$. These conditions specify an eigenvalue problem for λ, which can be solved numerically. The corresponding solution for $T(t)$ is

$$\frac{1}{T(t)} = -\lambda t + \frac{1}{T(0)}. \tag{6.39}$$

The results for the profile $w(x,t)/w_{max}(t)$ and the decay of the amplitude $1/w_{max}(t)$ are shown in figs. 8 and 9 in [2]. One can see that the agreement with the theoretical predictions is quite good.

The outer solution $u_{\text{outer}}(x,t)$ thus has the limits

$$u_{\text{outer}}(x,t) \to \pm C(t) \text{ when } x \to \pm 0. \tag{6.40}$$

We have thus computed the outer solution for the singularity. Now, we shall identify the inner solution.

6.6.2 Inner solution and asymptotic matching

We use the ansatz (6.13.1) for the inner solution. The resulting equation for the self-similar function $f(\xi)$ is

$$-\alpha f - \beta \xi f' = (f^2)''', \tag{6.41}$$

where

$$\alpha = -1 - 3\beta \tag{6.42}$$

and the prime denotes a derivative with respect to ξ. The initial conditions for the function f are $f(0) = 0 = f''(0) = 0$ since the function f is assumed to be an asymmetric function of ξ. By rescaling f and ξ in (6.41), one can always achieve $f'(0) = -1$. This initial-value problem can then be solved numerically. We thus obtain a one-parameter family of solutions $f_\alpha(\xi)$. The question is which value of α do we have to chose? The answer to this question is provided by analysing the behaviour of the function $f_\alpha(\xi)$ when $\xi \to \infty$.

Since the outer solution tends to a constant when $x \to 0$ due to (6.40), the inner solution must tend to the same constant when $\xi \to \infty$. It is proved in [2] that the function $f(\xi)$ for large ξ behaves as

$$f(\xi) \sim \xi^{-\alpha/\beta} + \text{subdominant terms} \tag{6.43}$$

The function $f(\xi)$ must tend to a constant in the limit $\xi \to \infty$. Thus, $\alpha = 0$, which corresponds to $\beta = -1/3$ due to (6.42).

This finishes the construction of the incomplete self-similar solution and the theoretical part of our paper. The rest of the paper [2] is devoted to the numerical simulations.

6.7 Numerical simulations

As we have mentioned, severe numerical difficulties have been encountered when simulating equation (6.33) on a computer, especially for small values of ν, see [6]. We have managed to construct a simple and robust numerical scheme, which can simulate (6.33) for arbitrary $\nu \geq 0$, even for $\nu = 0$. In the latter case, the shocks and their decay is accurately reproduced. The scheme bears the flavour of the upwind numerical schemes and is described in section 5.1 in [2]. We would like to attract the attention of physicists working with long-term computer simulations of (6.32) to this scheme.

To observe the approach to the singularity, we have written a fairly sophisticated fully implicit scheme, but we failed to achieve results good enough to confirm our theory of an incomplete similarity solution close to a singularity. Our implicit method, and the results obtained with its help, are described in section 5.2 of the article [2].

6.8 Conclusion

This paper has been rather pedagogical to us, since the results we have derived have taught us about the methods of

1. proving rigorous bounds for the solutions of PDEs,
2. performing asymptotic analysis of the singularities of PDEs,
3. numerical schemes, particularly upwind methods and implicit methods.

The singularity appearing in the equation is of the self-similar type. The aftermath of this singularity is a shock. Thus, another example has been added to the large number of PDEs exhibiting singularities of the self-similar type and shocks. Can there be more sophisticated singularities of some other functional form and what is the method for describing them? This is the question we would like to return to one day in the future.

6.9 Appendix: The initial condition for the self-similar solutions of the heat-conduction equation

Suppose, for example, that the initial condition for (6.7) is a point source $u_0(x) = \delta(x)$. Since $u(x,t) = t^\alpha f(xt^\beta)$, the function f must satisfy:

$$\lim_{t\to 0} t^\alpha \int f(tx^\beta)\mathrm{d}x = \delta(x). \tag{6.44}$$

We shall see that (6.44) determines the value of α (remember that for (6.7) $\beta = -1/2$ independent of α). It is easier to use the definition of the delta-function as a generalised functional. If we take an arbitrary infinitely-smooth function $\phi(x)$ with a compact support, then from the definition of the delta-function [5] we have

$$\delta(x) \circ \phi(x) = \phi(0), \tag{6.45}$$

which, combined with (6.44), becomes

$$\phi(0) = \lim_{t\to 0} t^\alpha \int f(tx^\beta)\phi(x)\mathrm{d}x. \tag{6.46}$$

Substitution of $\xi = xt^\beta$ transforms (6.46) into

$$\phi(0) = \lim_{t\to 0} t^{\alpha-\beta} \int f(\xi)\phi(\xi t^{-\beta})\mathrm{d}\xi =$$

$$\lim_{t\to 0} t^{\alpha-\beta} \int f(\xi)\left(\phi(0) + \xi t^{-\beta}\phi'(0) + \dots\right)\mathrm{d}\xi,$$

which can only be satisfied if

$$\begin{aligned} &\alpha = \beta \text{ and} \\ &\textstyle\int f(\xi)\mathrm{d}\xi = 1. \end{aligned} \tag{6.47}$$

Equation (6.19) can be integrated exactly in the case $\alpha = \beta = -1/2$. The general solution, satisfying the boundary conditions $u(x = \pm\infty, t) = 0$ is

$$f(x) = Ce^{-\xi^2/4}, \tag{6.48}$$

and the condition (6.47.2) gives $C = 1/(2\sqrt{\pi})$. One can see that the self-similar solution obtained by this method coincides precisely with the exact solution (6.16).

There is a one-parameter family of solutions $f_\alpha(\xi)$, given by (6.20). All these solutions satisfy the requirements $f \to 0$ when $\xi \to \pm\infty$, so there is no restriction on α due to asymptotic matching. All the initial conditions, leading to self-similar solutions, are given by the following one-parameter family of generalised functions

$$u_0(x,\alpha) = \lim_{t\to 0} t^\alpha f_\alpha(xt^{-1/2}). \tag{6.49}$$

One can express $u_0(x,\alpha)$ in terms of the generalised derivative of the delta-function

$$u_0(x,\alpha) = C\delta(x)^{(p)} = C\int_{-\infty}^{+\infty}(-ik)^p e^{-ikx}.\mathrm{d}k \tag{6.50}$$

Here C is an arbitrary constant and p is the order of the derivative, which is, in general, an arbitrary real number (not necessarily an integer).

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