Global Estimates and Shocks for the Noiseless Conserved Kardar-Parisi-Zhang Equation

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Abstract

We present a series of results concerning the equation $u_t = (u_x)^2_{xx} - \nu u_{xxxx}$, which describes non-equilibrium surface dynamics in the presence of volume conservation. We investigate whether a singularity in the solution can be formed in finite time, as was reported recently in the literature. For $\nu = 0$ we derive several integrals of motion, which allow us to prove that u always remains bounded, even if a singularity is formed. For $\nu > 0$ we prove that the L_2 norm of the solution decays at least exponentially, on a time scale of order L^4/ν, where L is the system size. Moreover we show that the solution remains bounded for all times if the L_2 norm of the derivative u_x is sufficiently small initially. Based on these results and on the numerical integration of the equation we conjecture that no singularities occur for $\nu > 0$. For $\nu = 0$ we predict wedge-like singularities to appear at local maxima, in which u_x changes sign, but u as well as $(u_x)^2$ remain continuous. We show that a self-similar approach to such singularities is possible and present some numerical evidence for this scenario. We are able to continue the simulations beyond the formation of singularities. The simulations show the formation of non-moving shocks (in u_x), which decay algebraically in time.

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1 Introduction

This paper is devoted to the study of the equation

$$u_t = (u_x)^2_{xx} - \nu u_{xxxx}, \tag{1}$$

$$u = u(x,t),\ -L/2 \le x \le L/2,\ u(t=0) = u_0(x),$$

where the subscript x or t denotes the partial derivative with respect to x or t and ν is a positive real number. The parameter ν is sometimes called the surface tension or *hyperviscosity*.

We consider two boundary conditions:
Periodic: $u(x,t)$ is a smooth periodic function in x with period L,

$$u(x,t) = u(x+L,t). \tag{2}$$

Rigid:

$$u(x=-L/2) = u_x(x=-L/2) = u(x=L/2) = u_x(x=L/2) = 0. \tag{3}$$

1.1 Motivation and physical background

Equation (1) was first written down by Sun, Guo and Grant [1] in a study of non-equilibrium surface dynamics in the presence of a conservation law for the volume enclosed by the 'height function' $u(x,t)$. It subsequently reappeared in the context of thin film growth by molecular-beam epitaxy (MBE), where the surface relaxation processes (mainly surface diffusion) can be assumed, to a good approximation, to be volume conserving [2, 3, 4].

This implies that the equation of motion for u takes the form of a local conservation law, $u_t + j_x = 0$ with a surface mass current j. Physically, one expects the current to be driven by the gradient of some generalized chemical potential $\mu(x,t)$, so that $j = -\mu_x$ and the evolution law for u becomes $u_t = \mu_{xx}$. This is precisely of the form (1) with a chemical potential

$$\mu(x,t) = (u_x)^2 - \nu u_{xx}. \tag{4}$$

The second, linear term in (4) reflects the well-known Gibbs-Thomson effect [5]: Atoms at local surface maxima ($u_{xx} < 0$) have less nearest neighbour bonds than at local minima ($u_{xx} > 0$), and consequently their chemical potential is enhanced relative to the equilibrium value $\mu = 0$; the coefficient ν is therefore proportional to the surface tension.

The first, nonlinear contribution to (4) arises only under nonequilibrium conditions [6], when an excess density $\rho_{\rm ex}$ of adsorbed atoms (= adatoms) is created by some external influence, such as a deposition flux [2] or an ion beam [1, 6] bombarding the surface. Since the local value of the excess density is determined by the balance between the creation of adatoms and their capture at surface steps, it will depend on the local step density or, equivalently, the local tilt u_x. If $u(x,t)$ measures the height of the surface relative to some high symmetry ('singular') reference plane, then the leading (smooth) tilt dependence is of the form $\rho_{\rm ex} = \rho_0 - \lambda(u_x)^2$ with $\lambda > 0$: the step density *increases*, hence the adatom density *decreases* upon tilting. At low densities the chemical potential is proportional to the excess density, hence the effect may equally well be represented by a similar contribution to μ; to arrive at the form (4) we have used units such that $\lambda = 1$ and additionally performed a reflection $u \to -u$. It should therefore be kept in mind that physically the phenomena which occur at the *maxima* of u in fact take place at the *minima* of the height function. We shall see in Section 4 that the physical interpretation of $(u_x)^2$ as a contribution to the local chemical potential is very helpful in guiding the intuition regarding the nature of possible singularities.

The studies mentioned above [1, 2, 4] have typically addressed the properties of (1) in the presence of stochastic forces modelling the shot noise in the deposition beam or thermal fluctuations. In contrast, the present work focuses on the *noiseless* equation. Our main motivation derives from a conjecture put forward by Tu [7] that, for sufficiently small $\nu > 0$, the noiseless equation may develop singularities where the height field u diverges in finite time. This conjecture was based on a numerical integration of (1), and it was used to explain the severe numerical difficulties encountered when integrating the *noisy* equation [7, 8].

To put the issue into perspective, we note the close relation between (1) and the noiseless Kardar-Parisi-Zhang (KPZ) equation [9]

$$u_t = \nu u_{xx} - (u_x)^2, \tag{5}$$

which reduces to (1) by placing the negative Laplace operator $-\partial^2/\partial x^2$ in front of the terms on the right hand side. Since the Laplacian ensures conservation of u, (1) is referred to as the conserved KPZ equation.

The behavior of the solutions to (5) is well understood, due to the existence of the Cole-Hopf transformation [10]

$$Z(x,t) = \exp[-\nu^{-1} u(x,t)] \tag{6}$$

which turns (5) into a *linear* diffusion equation for Z, with diffusion coefficient ν. This implies, in particular, that smooth initial data remain smooth as long as $\nu > 0$. In the 'inviscid' limit $\nu \to 0$ finite time singularities, in the form of discontinuities of the first derivative, develop at the maxima of u. These are just the familiar shock waves of Burgers' equation [10]

$$w_t + (w^2)_x = \nu w_{xx} \tag{7}$$

which governs the dynamics of $w = u_x$.

The simplification afforded by the linearising transformation (6) is not available for the conserved equation (1). The goal of this work will therefore be to obtain global bounds on the solutions of (1), as well as its inviscid limit

$$u_t = (u_x)^2_{xx}, \tag{8}$$

and the viscous and inviscid 'conserved Burgers equation',

$$w_t = (w)^2_{xxx} - \nu w_{xxxx}, \tag{9}$$

and

$$w_t = (w)^2_{xxx}, \tag{10}$$

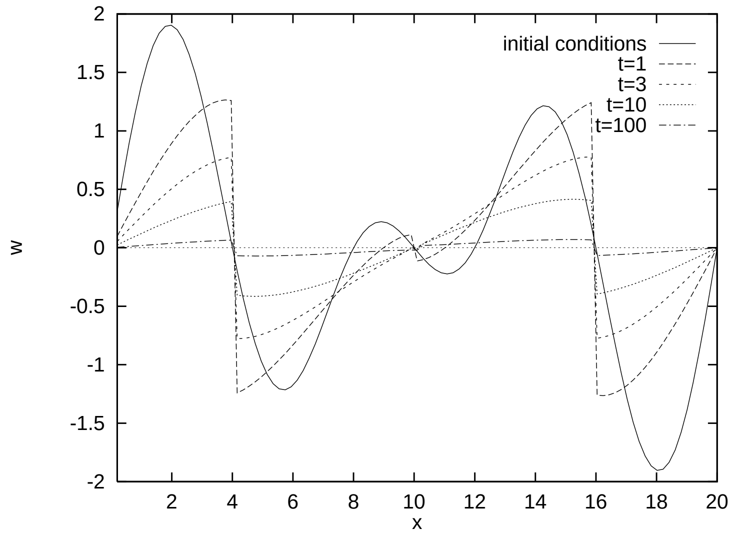

Figure 1: Numerical solution of the inviscid w equation (10). One can clearly see the formation of shocks at the maxima of $u = \int w dx$ predicted by our theory. The shocks finally decay, remaining sharp, but they do not move. Periodic boundary conditions, $L = 20$.

where, again, $w = u_x$. Equations related to (10) have recently been shown to possess soliton-like solutions with compact support ('compactons') [11].

For the inviscid case which, heuristically, would be expected to be the most singular, we are able to prove that all solutions remain bounded. This rigorously rules out the scenario suggested by Tu [7]. The proof does not extend globally to the case $\nu > 0$, but based on the somewhat weaker results stated in Theorem 2 below, and supported by the numerical simulations described in Section 5, we conjecture that the solutions remain bounded in this case also.

Note that, since the "total mass" $\int u(x,t)dx$ in the equation (1) and (8) is conserved, we shall perceive $u(x,t)$ not as the absolute height of the molecular layer (which, of course, always has to be a non-negative number), but rather as a *deviation* from the average height. That is, we shall put $\int u(x,t)dx = 0$. This does not change the physical essence of the equations, but is more convienent from the technical point of view.

The strong analogy between equations (9) and (7) is obvious from the numerical simulations of (9) shown in figure 1 and figure 2, made for $\nu = 0$ and $\nu = 0.1$ respectively.

For $\nu = 0$ it is seen that the curve $w = u_x$ around the zeroes, corresponding to maxima of u, steepens and finally forms a discontinuity (shock). Like in the Burgers case, the shocks decay, although they do not move. The formation and decay of shocks will be discussed in detail in Section 4.

For $\nu > 0$ (figure 2) the shocks never become sharp and the solution remains smooth, just like in the Burgers case.

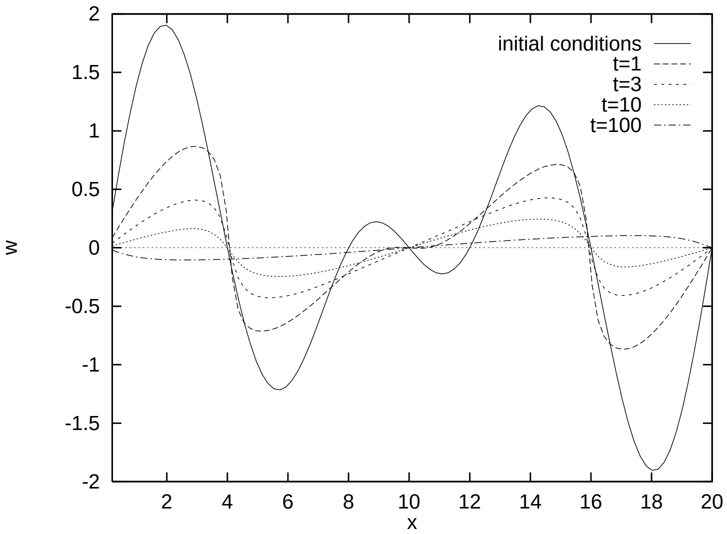

Figure 2: Numerical solution of the w equation (9) with $\nu = 0.1$. In this case, the shocks are smoothed out by the viscosity. No singularity is formed. Periodic boundary conditions, $L = 20$.

1.2 Assumptions

To be able to proceed, we shall need the smoothness assumption for the equations (1), (8), (9) and (10). To our knowledge, no theorems about the existence and uniqueness of solutions to these equations for smooth initial conditions have been proved. Therefore, we will have to make the basic assumptions of this paper:

Assumption 1 *Let $u(x,t)$ be the solution of equation (8) with the initial conditions $u(x,t=0) = u_0(x)$. For each $u_0 \in C_x^4[-L/2, L/2]$ there exists a time $T(u_0)$ such that $u \in C_x^4[-L/2, L/2] \times C_t^1[0, T(u_0)]$.*

Here $C_x^4[-L/2, L/2]$ denotes the space of functions, posessing 4 continuous partial derivatives with respect to x and $C_t^1[0, T(u_0)]$ is the space of functions, having a continuous partial derivative with respect to t. Correspondingly, we make a similar smoothness assumptions for the solutions of the equations (1), (9) and (10). For the inviscid w equation (10) we assume that the solution is C^3 smooth in x. For equation (1) we require the solution to possess 5 continuous derivatives in x and, correspondingly, the solution of the viscous w equation (9) is assumed to be 4 times continuously differentiable in x.

From now on, this Smoothness Assumption will be implicitly assumed in the statements of all the Theorems and Lemmas.

One might wonder whether the Assumption 1 is reasonable. Consider, for example equation (10). If we start with some oscillating initial conditions $w_0(x)$, there will be points in space where w changes sign. Hence, the coefficient of the highest derivative changes sign, which can lead to loss of smoothness. We shall argue in what follows that the points x_0 where the solution $w(x_0)$ changes sign, corresponding to the maxima in $u(x,t)$, are in fact the "dangerous" points.

However, the loss of smoothness occurs *not immediately*, but after some time T, which is dependent on the initial conditions. This is confirmed by our numerical simulations, where the solution remains seemingly smooth for a well-defined period of time.

2 Rigorous results for $\nu = 0$.

In this Section, we shall not consider all initial conditions, but only the generic ones. We make a genericity assumption about the initial conditions $u_0(x)$.

Assumption 2 *Suppose x_0 is an extremum of the function $u_0'(x)$. Then, $u_0'(x_0) \neq 0$.*

2.1 Behaviour of local extrema

Lemma 1

1. *The number of maxima and the number of minima of the solution of (8) is conserved. Moreover, a maximum cannot become a minimum and vice versa.*
2. *The position of maxima and minima changes differentiably with respect to t for a generic initial condition.*
3. *Suppose we start with a generic initial condition. Let $a(t)$ and $b(t)$ denote the position of two subsequent extrema of the solution $u(x,t)$, that is, two subsequent zeroes of the first derivative $u_x(x,t)$. Then the following integral is conserved:*

$$\int_a^b u_x(x,t)^3 \, dx = const. \tag{11}$$

Part 1.

Maxima and minima of u occur when $w = u_x = 0$. New maxima and minima appear or disappear when an extremum of $w(x)$ crosses the line $w = 0$. Denote the position of the local extremum of $w(x,t)$ by $x_0(t)$. We expand $w(x,t)$ locally around $x_0(t)$ in power series up to the third order in $(x - x_0)$:

$$w(x,t) = a(t) + b(t)(x - x_0)^2 + c(t)(x - x_0)^3 + o((x - x_0)^3).$$

This is possible according to the Smoothness Assumption. Inserting this expansion in (10) and putting $x = x_0$ gives

$$\dot{a} = 12a(t)c(t). \tag{12}$$

It is clear that the change of the sign of $a(t)$ is a necessary condition for two new extrema to be born. Because the initial condition was assumed to be generic (Assumption 2), $a(0) \neq 0$. It is obvious, however, from the structure of the equation (12) that $a(t)$ does not changes sign at any finite time (provided that the coefficient $c(t)$ is bounded, that is the function $w(x,t)$ is smooth). Therefore, new maxima and minima can never appear or disappear and their total number is conserved.

It is evident from continuity that a maximum cannot become a minimum, and vice versa, without the creation of additional extrema, which is not allowed because of the previous statement. Part 1 of Lemma 1 is proved.

Part 2

First, we show that $a(t)$ is continuous. Suppose that $a(t)$ is an extremum of the solution $u(x,t)$. Then, $a(t)$ is given by the implicit relation:

$$w(a(t),t) = 0. \tag{13}$$

The mapping $w(x,t)$ is a locally monotonic and continuous function of x around the point $x = a(t)$ for any value of the parameter t such that the Smoothness Assumption is satisfied. In fact, $x = a(t)$ cannot become an extremum (in x) of the function w according to Part 1 of this Lemma, since $x = a(0)$ was not an extremum initially. Thus, the inverse mapping $w \to x(w,t)$ is also locally continuous (and monotonic) in w around the point $x = a(t)$, i.e $w = 0$. Since $w(x,t)$ changes continuously with t, $a(t)$ will be continuous, because $a(t) = x(w = 0, t)$. Next, we show that $a'(t)$ is continuous. By the Smoothness Assumption and the Implicit Function Theorem [12], we have that $a(t)$ is differentiable provided that $w_x(a(t),t) \neq 0$. The time derivative $a'(t)$ is then equal to

$$a'(t) = -\left.\frac{w_t}{w_x}\right|_{x=a(t)} = -\left.\frac{(w^2)_{xxx}}{w_x}\right|_{x=a(t)} = -6w_{xx}|_{x=a(t)}\,. \tag{14}$$

The problems may arise at a time instant t_0 such that $w_x(a(t_0),t_0) = 0$.

From (14) and the Smoothness Assumption, we deduce that $a'(t)$ is continuous in a sufficiently small neighbourhood of t_0 with t_0 excluded: $a(t) \in C^1[(t_0-\epsilon,t_0)\cup(t_0,t_0+\epsilon)]$ for sufficiently small ϵ and the limits $\lim_{t\nearrow t_0} a'(t)$ and $\lim_{t\searrow t_0} a'(t)$ coincide. Also, $a(t)$ is continuous at the point t_0. Thus, $a(t)$ is continuosly-differentiable at $t = t_0$.

Part 3.

Let $a(t)$, $b(t)$ denote two *subsequent* zeroes of the first derivative of the solution:

$$w(a(t),t) = w(b(t),t) = 0,\ w(x,t) \neq 0,\ x \in (a(t),b(t)), \tag{15}$$

where $w(x,t)$ is a solution of (10). Then, using equation (10) and performing several integrations by parts,

$$\frac{d}{dt}\int_{a(t)}^{b(t)} w(x,t)^3 \mathrm{d}x =$$

$$3\int_{a(t)}^{b(t)} w^2(w^2)_{xxx}\mathrm{d}x + b'(t)w^3\,|_{x=b} - a'(t)w^3\,|_{x=a} =$$

$$-3\int_{a(t)}^{b(t)} (w^2)_x(w^2)_{xx}\mathrm{d}x + 3w^2(w^2)_{xx}\,\Big|_a^b =$$

$$-\frac{3}{2}\left((w^2)_x\right)^2\Big|_a^b = -\frac{3}{2}(2ww_x)^2\Big|_a^b = -6(ww_x)^2\Big|_a^b = 0.$$

The integrated terms and the terms of the type $a'(t)w^3\,|_{x=a}$ vanish because of the condition (15). The derivatives of $a(t)$ and $b(t)$ can be taken according to part 2.

This concludes the proof of Lemma 1. Let us now suppose that we start with some oscillating initial condition, that is we have more than one extremum of the function $u(x,t)$. Let us further suppose that around the local extremum $x_0(t)$ the solution $u = u(x,t)$ is represented by

$$u(x,t) = a(t) + b(t)(x - x_0)^2 + O((x - x_0)^3).$$

The equation for the evolution of $a(t)$ will then be

$$\dot{a} = 8b^2 \geq 0. \tag{16}$$

Therefore, the height at the extremum moves up with time. Recall, however, that the total average of $u(x,t)$ with respect to x has to be conserved. As usual we set this average to zero:

$$\int u(x,t)\mathrm{d}x = 0.$$

Assume also that the initial condition $u(x, t = 0)$ takes positive values at maxima and negative at minima (for example, $u(x, t = 0) = \sin(2\pi x/L)$). Then the tips of the maxima move away from the line $u = 0$, and the bottoms of the minima move closer to this line. This suggests the following scenario for the formation of a singularity: the negative parts become increasingly flat whereas the positive parts of the solution become more "narrow" to maintain the zero average of the solution, at least initially. This is, of course, not a rigorous statement. However, it shows that a maximum of the solution is a plausible place for the formation of a singularity. This is well justified by the numerical simulations, where we see the formation of a singularity from a maximum, rather then from a minimum; additional arguments will be given in Section 4.

It is clear form the structure of the equation (8) that if we start from a symmetric initial condition $u(x, t = 0) = u(-x, t = 0)$, the solution always remains symmetric, i.e. $u(x,t) = u(-x,t)$ (we do not discuss the stability of such a solution here). For such a solution there is always an extremum at $x = 0$. The question is whether this extremum will be global. Lemma 1 Part 1 gives us the answer to this question in the case when the initial condition has only one maximum at this point.

Lemma 2 *Suppose we start with a symmetric initial condition*

$$u(x,0) = u(-x,0),$$

with the point $x = 0$ being the unique global maximum. Then $x = 0$ will remain the global maximum for all t, for which the Smoothness Assumption for the solution of (8) holds.

Proof.

It is easy to see that a symmetric ($x \to -x$) initial condition will give rise to a symmetric solution of (8). Therefore $x = 0$ will always be an extremum.

According to Lemma 1 Part 1, the solution will always have only one maximum. This means that $x = 0$ will always be the global maximum for the solution. The Lemma is proved.

Given our conclusion that a singularity forms from a maximum rather than form a minimum, an immediate consequence of Lemma 2 is that, if the initial conditions have the form of a single peak (e.g. a Gaussian), there is only one place we can expect to get a singularity, namely at $x = 0$.

2.2 Global bounds

Now, we would like to establish the global bounds for the inviscid solution. To do that, we need to define the functional space $\mathcal{S}$ we are going to work in. We consider the phase space $\mathcal{S}$ to be

$$\begin{array}{c}\mathcal{S} = \text{closure in the space } W_3^1 \text{ of the set of functions} \\ f(x) \text{ with } \int f(x)\mathrm{d}x = 0.\end{array} \tag{17}$$

We shall equip this phase space with the norm

$$\|u(\cdot,t)\|_{\mathcal{S}} = \left(\int |u_x(x,t)|^3 \mathrm{d}x\right)^{1/3}. \tag{18}$$

One can show [12] that in the phase space $\mathcal{S}$, the norm (18) is equivalent to W_3^1 norm

$$\|u(\cdot,t)\|_{W_3^1} = \left(\int |u(x,t)|^3 \mathrm{d}x\right)^{1/3} + \left(\int |u_x(x,t)|^3 \mathrm{d}x\right)^{1/3}. \tag{19}$$

If the restriction $\int u(x,t)\mathrm{d}x = 0$ is removed, the norm (18) becomes only a pseudo-norm. However, since the average of u is conserved (see Lemma 1), this is not a real restriction on the solution, as the $u(x,t)$ denotes the *deviation* from the average height.

Now we obtain the global bounds for the solution of (8):

Theorem 1 *($\nu = 0$)*

1. *The integrals $\int u dx$ and $\int u^2 dx$ are constants of motion of (8).*
2. *The integral*

   $$\int |u_x(x,t)|^3 dx,$$

   and, therefore, the $\|u(\cdot,t)\|_{\mathcal{S}}$ norm of the solution, is conserved.
3. *The $C-$norm of the solution is bounded, i.e*

   $$sup_x |u(x,t)| \leq C_1$$

   where the constant C_1 depends on the initial conditions and the size of the system.

Part 1. The proof of this part is trivial. Integrate the equation (8) with respect to x from $-L/2$ to $L/2$. Then,

$$\frac{d}{dt}\int u(x,t)\mathrm{d}x = 2u_x u_{xx}\Big|_{-L/2}^{L/2} = 0$$

because of the boundary conditions (2) or (3). In a similar way, we multiply the equation (8) by $u(x,t)$, integrate and obtain the second conservation law

$$\frac{d}{dt}\int u^2(x,t)\mathrm{d}x = 4uu_x u_{xx}\Big|_{-L/2}^{L/2} - \frac{4}{3}u_x^3\Big|_{-L/2}^{L/2} = 0.$$

Part 2. Let $w(x,t)$ be a solution of the equation (10). Let us enumerate all the zeroes of $w(x,t)$ in increasing order:

$$w(a_i(t),t) = 0,$$

$$a_1(t) < a_2(t) < \ldots < a_n(t).$$

According to Lemma 1, the integral

$$\int_{a_{i-1}}^{a_i} w^3\mathrm{d}x$$

is conserved. But inside the interval $[a_{i-1}, a_i]$ the function w has a constant sign. Therefore, the integral

$$\int_{a_{i-1}}^{a_i} sign(w)w^3\mathrm{d}x = \int_{a_{i-1}}^{a_i} |w^3|\mathrm{d}x$$

is also conserved. Performing the summation over all the intervals, we obtain the desired conservation law:

$$\int_{-L/2}^{L/2} |w^3|\mathrm{d}x = const.$$

Here we have used the results of Lemma 1. The number of terms in the sum is constant according to Part 1, the numbers $a_i(t)$ preserve their ordering for all time t such that the Smoothness Assumption holds and their amount remains constant in time, since no new extrema can be born. The functions $a_i(t)$ are differentiable with respect to t according to Part 2 of Lemma 1. Each term in the sum is constant according to Part 3 of Lemma 1.

Note. Obviously, the integral

$$\int_{-L/2}^{L/2} w(x,t)^3\mathrm{d}x$$

(the integration now is performed over the whole x-space) is conserved along with the integrals derived above. However, this integral is not always informative. For example, for the symmetric solutions $u(x,t) = u(-x,t)$ we have $w(x,t) = -w(-x,t)$ and this integral is identically zero.

Part 3. We consider the integral of motion for (10)

$$\int_{-L/2}^{L/2} |w^3| \mathrm{d}x = a = const.$$

This is nothing but the third power of the norm of the solution of (8) in the space $\mathcal{S}$. This norm, as we have emphasised, is equivalent to the W_3^1 Sobolev norm. According to the Sobolev embedding theorems (see formula (65) in the Appendix), it bounds the sup-norm (C-norm) of the solution: There exists a positive constant K such that

$$\|u(\cdot,t)\|_C \leq K\|u(\cdot,t)\|_{W_3^1} = K\left(\int_{-L/2}^{L/2} |u_x(x,t)|^3 \mathrm{d}x\right)^{1/3} = Ka^{1/3} = C_1.$$

Here we apply the theorem (65) for the case $p = 3$, $l = 1$ and the dimension of the space $n = 1$.

This concludes the proof of Theorem 1.

The bound in the norm $\|\cdot\|_{\mathcal{S}}$ actually puts stronger restrictions on the solutions than just the bound of the C-norm. If, e.g, $u(x)$ has a singularity of the form $u \sim$(regular part)$+(x-x_0)^\alpha$, then $\alpha > 2/3$.

3 Rigorous results for $\nu > 0$.

Next we turn our attention to the equation (1). The strategy employed in proving the boundedness of $sup_x|u(x,t)|$ for the inviscid case cannot be used here, because it relies on the conservation of the number of extrema stated in Lemma 1, which breaks down for $\nu > 0$. Indeed, it is well known [5, 13] that the fundamental solution of the fourth order heat equation $u_t = -\nu u_{xxxx}$ has an infinite number of changes of sign in its exponential tail. We can nevertheless derive a global bound for the L_2-norm of the solution by showing that the integral

$$\int u(x,t)^2 \mathrm{d}x$$

decays exponentially for all initial conditions.

If the initial condition is small enough, we can derive a stronger estimate: We can show that the functional

$$\int u_x(x,t)^2 \mathrm{d}x$$

is a Lyapunov function and will decay exponentially. From this, it will follow that $sup_x|u|$ decays exponentially also. In all cases the lifetime characterising the exponential decay is of the order L^4/ν, which is just the characteristic time scale for the *linear* equation $u_t + \nu u_{xxxx} = 0$ [5].

These results are summarised in the following theorem.

Theorem 2 *($\nu > 0$)*

1. *The (square of the)* L_2*-norm of the solution, i.e*

$$\|u(\cdot,t)\|_2 \equiv \int u(x,t)^2\,dx,$$

is a Lyapunov function, meaning that

$$\frac{d}{dt}\int u(x,t)^2\,dx \leq 0.$$

2. *For the periodic boundary conditions (2) with* $\int u(x,t)\,dx = 0$ *and the rigid boundary conditions (3) the* L_2*-norm of the solution is decreasing at least exponentially:*

$$\|u(\cdot,t)\|_2 \leq C_2 \exp\left(-C_3 t\right)$$

for some $C_2, C_3 > 0$*. This is true for all initial conditions.*

3. *For the periodic boundary conditions (2), if the initial conditions satisfy*

$$\int u_x(x,t=0)^2\,dx < C_4, \tag{20}$$

then the following estimate holds:

$$\int u_x(x,t)^2 \leq C_5 \exp\left(-C_6 t\right) \tag{21}$$

for some positive constants C_4*,* C_5*,* C_6*, which depend only on the value of* ν *and the size of the system* L*.*

4. *Under the conditions of part 3,*

$$sup_x|u(x,t)| \leq C_7 \exp\left(-C_6 t\right),$$

where $C_7 > 0$ *is a constant and* C_6 *is the same as in (21).*

Part 1.

The proof is straightforward. If we multiply the equation (1) by u and integrate with respect to x, then due to the boundary conditions (2) or (3) the non-linear terms integrate out and we get

$$\frac{d}{dt}\int (u(x,t))^2 \mathrm{d}x = -2\nu \int (u_{xx}(x))^2 \mathrm{d}x \leq 0. \tag{22}$$

Therefore, the L_2 norm of the solution always remains bounded.

Part 2. We make repeated use of the Poincaré inequality (63) to relate the integral over $(u_{xx})^2$, appearing on the right hand side of (22), to the integral over u^2 on the left:

$$\int (u_{xx})^2 \mathrm{d}x \geq L^{-2}\int (u_x)^2 \mathrm{d}x \geq L^{-4}\int u^2 \mathrm{d}x. \tag{23}$$

Therefore

$$\frac{d}{dt}\int (u(x,t))^2 \mathrm{d}x \le -\frac{2\nu}{L^4}\int (u(x))^2 \mathrm{d}x. \tag{24}$$

Since the L_2-norm of the solution is non-negative, from the inequality (24) we conclude that

$$\|u(\cdot,t)\|_2 \le \|u(\cdot,t=0)\|_2 \exp(-L^{-4}\nu t). \tag{25}$$

Note. For the physically interesting regimes $L \gg 1$ and $\nu \ll 1$, the life-time for the system is actually very large (of order L^4/ν). Therefore, it is very hard to detect this decay numerically.

Part 3.

We multiply the equation (9) with w and integrate from $-L/2$ to $L/2$ with respect to x to obtain the evolution equation for the "kinetic energy" $\int w^2 \mathrm{d}x$:

$$\frac{d}{dt}\int (w(x,t))^2 \mathrm{d}x = -2\int (w_x)^3 \mathrm{d}x - 2\nu \int (w_{xx})^2 \mathrm{d}x. \tag{26}$$

We estimate the right-hand side in (26) as follows:

$$-2\int (w_x)^3 \mathrm{d}x - 2\nu\int (w_{xx})^2 \mathrm{d}x$$

$$\le 2\int |w_x|^3 \mathrm{d}x - 2\nu\int (w_{xx})^2 \mathrm{d}x \le 2 sup_x |w_x| \int (w_x)^2 \mathrm{d}x - 2\nu \int (w_{xx})^2 \mathrm{d}x$$

$$\le 2\sqrt{L}\left(\int (w_{xx})^2 \mathrm{d}x\right)^{1/2}\int (w_x)^2 \mathrm{d}x - 2\nu\int (w_{xx})^2 \mathrm{d}x,$$

the last inequality following from the Poincaré lemma (62).

It is clear that the following inequality holds:

$$\int w_x^2 \mathrm{d}x \le \left(\int w^2 \mathrm{d}x\right)^{1/2}\left(\int w_{xx}^2 \mathrm{d}x\right)^{1/2}. \tag{27}$$

In fact, if we integrate the left-hand side of (27) by parts and use the Schwartz inequality, we get (27). Due to (27), we have the following estimate:

$$\frac{d}{dt}\int (w(x,t))^2 \mathrm{d}x \le 2\left\{\sqrt{L}\left(\int (w(x,t))^2 \mathrm{d}x\right)^{1/2} - \nu\right\}\int (w_{xx})^2 \mathrm{d}x$$

We see that if the initial conditions satisfy

$$\left(\int (w(x,t=0))^2 \mathrm{d}x\right)^{1/2} \le \frac{\nu}{2\sqrt{L}},$$

then the integral

$$\int (w(x,t))^2 \mathrm{d}x$$

is a Lyapunov function. Therefore, for all t we have

$$\frac{d}{dt}\int (w(x,t))^2 \mathrm{d}x \le -\nu\int (w_{xx})^2 \mathrm{d}x.$$

The rest of the proof follows by the repeated application of the Poincaré lemma (62) and is completely equivalent to the proof of the Theorem 2 Part 2.

Part 4. Part 4 is now a direct corollary of Part 3 of Theorem 2 and Poincaré lemma (62).

Theorem 2 is proved.

We shall note that Part 3 of Theorem 2 sets in fact stronger bounds than just the boundedness of the sup norm of the solution $u(x,t)$. If a solution forms a cusp, say $u(x,t) =$(regular part)$+(x-x_0)^\alpha$, then this cusp should be sufficiently weak, namely $\alpha > 1/2$.

4 The Structure of the Singularity for $\nu = 0$

We have argued above in Section 2.1 that in the inviscid case $u(x,t)$ can only increase at local maxima, which leads to a sharpening of the peak and eventually to the formation of a singularity. It is useful to restate the argument in terms of the chemical potential $\mu = (u_x)^2$ introduced in equation (4). Let x_0 denote the position of the maximum. As long as u remains smooth at $x = x_0$, $\mu(x_0) = 0$, and since $\mu \geq 0$ everywhere, it follows that μ has a *minimum* at $x = x_0$. The mass current $j = -\mu_x$ then flows into the maximum of u, which continues to move upward. Analogously, one can easily see that the mass current flows into the minima, filling them up. This confirms our guess stated in Section 2.1, that the peaks become narrow and the valleys flatten out, thus leading to a singularity at the maxima. On physical grounds, one expects however that the dynamics will eventually succeed in relaxing the chemical potential differences even for $\nu = 0$. This requires that x_0 becomes a *maximum* of μ, which from the preceding discussion is possible only if a singularity forms. More precisely, in order to have a finite, positive value $\mu_0 = \mu(x_0) > 0$ after the formation of the singularity, the height profile has to attain a "wedge" shape

$$u(x,t) = A + a_0|x - x_0| + \dots \tag{28}$$

with $\mu_0 = a_0^2$, which subsequently may start to decay. Furthermore, for the decay rate $-u_t(x = x_0)$ to be nonzero and bounded, the chemical potential should possess at least two spatial derivatives and $\mu_{xx}(x = x_0) < 0$.

A similar scenario has been established for the evolution equation

$$u_t = (u_x)^2 u_{xx} \tag{29}$$

which arises in the context of surface dynamics below the roughening transition [14]. In this case the requirement that the right hand side of the equation remains finite but nonzero beyond the formation of the singularity implies a behavior $u \sim |x - x_0|^{4/3}$ close to the maximum. In the following we corroborate our physical picture for equation (8).

In figure 1 we show the result of our numerical simulation of the inviscid w equation (10), passing through the singularity. The detailed discussion of the numerics is given in Section 5. The initial conditions were $w_0(x) = \sin(4\pi x/L) + \sin(6\pi x/L)$, where L is the length of the system and periodic boundary conditions were used. The zeroes of the solution $w(x,t)$ correspond to the extrema of the function $u(x,t)$, where $u_x = w$. One can see that the singularity is formed at the maxima of u, in accordance with the previous discussion. The singularity in w is a shock

$$w(x,t) = a_0 \mathrm{sgn}(x - x_0) + b_1(x - x_0) + a_1|x - x_0| + \dots \tag{30}$$

which corresponds precisely to the wedge (28). Here a_k are coefficients at the singular terms and b_k are coefficients at the regular terms. To make contact with the standard analysis of shocks in nonlinear conservation laws [15], we rewrite (10) in the form

$$w_t + F_x = 0 \tag{31}$$

using the flux $F = -(w^2)_{xx} = -\mu_{xx}$. Across the shock, the flux F may be discontinuous, however, it should not diverge, since it is a physically relevant quantity (namely, the decay rate $-w_t$ of the surface profile). Thus, w^2 must at least be continuous across the singularity. One can see that the form (30) satisfies this requirement:

$$w^2 = a_0^2 + c_1(x - x_0) + d_1|x - x_0| + \dots \tag{32}$$

which is continuous at the point x_0. The coefficients c_k (regular terms) and d_k (singular terms) can be expressed in terms of the coefficients a_k and b_k from (30). The boundedness of the flux requires $(w^2)_x$ to be continuous, thus the coefficient $d_1 = 2a_0b_1$ has to vanish, and since we assume that $a_0 \neq 0$, we must have $b_1 = 0$. An infinite series of evolution equations for the coefficients a_k and b_k can be constructed by substituting the expression (30) into equation (10).

The motion of the shock is governed by the difference of the flux to the left and to the right of the singularity. Indeed, suppose $x_0(t)$ is the position of the singularity. By integrating equation (31) from the point $x_0(t) - \epsilon$ to $x_0(t) + \epsilon$ and letting $\epsilon \to 0$, we obtain an evolution equation for the shock position

$$\dot{x_0} = \frac{[F]}{[w]} \tag{33}$$

with the standard notation $[y] = y(x_0 + 0) - y(x_0 - 0)$. Now remember that w changes sign across the singularity, so $[w] = 2w(x_0 + 0)$. Thus, the expression (33) transforms to

$$\dot{x_0} = \frac{[w^2_{xx}]}{2w(x_0 + 0)}. \tag{34}$$

If w^2_{xx} remains continuous across the shock, the velocity of the shock is zero. We can see that this is the case for our numerical simulations: the shocks do not seem to move at all. We have not been able to find initial conditions leading to moving shocks, although we cannot exclude their existence.

We shall also emphasise that the number of zeroes of w, i.e., the number of extrema of u is *not conserved* after the singularity has been formed - Lemma 1 is violated, as one can easily see from figure 1. The three zeroes in the middle of the picture collapse to one after a certain time, when the small shock disappears.

4.1 Self-similar approach to the singularity

We now investigate how a singularity of the form (30) can appear from smooth initial data in finite time. We use the differentiated version (10) (where, again, $w(x,t) = u_x(x,t)$) and look for a scale-invariant solution in the form

$$w(x,t) = (t_0 - t)^{\alpha} f(x(t_0 - t)^{\beta}), \tag{35}$$

where t_0 is the time at which the singularity is formed. We hope that this is a good approximation to the solution of (10) in some region close to $x = 0$ when t approaches t_0. Since this ansatz is supposed to become singular at $t = t_0$, we have $\beta < 0$. Further, since we hope to describe the formation of a singularity of the type (30), w should not diverge, hence $\alpha \geq 0$. Substitution of (35) into (10) gives $\alpha = -3\beta - 1$. The scaling function f satisfies

$$-\alpha f - \beta \xi f' = (f^2)''', \tag{36}$$

where $\xi = x(t_0 - t)^{\beta}$ and the prime denotes the derivative with respect to ξ. The choice of β is governed by the asymptotics of (36) when $\xi \to \infty$. The conditions at infinity are $f \to \pm a_0$ when $\xi \to \pm\infty$, since this will give us a transition to the singularity assumed by the ansatz (30). The self-similar solution $w(x,t)$ with the parameters α and β satisfying these requirements is referred to as a self-similar solution of the second kind. The method for the selection of α and β from the far-field behaviour of the function f is described, for example, in the book [16], chapter 12.3.

We focus here on the case where u is an even function with a single maximum at $x = 0$; then $f(\xi)$ has to be an odd function of ξ. Hence, the initial conditions for $f(\xi)$ at $\xi = 0$ are $f(0) = 0$ and $f''(0) = 0$. Also, since $x = 0$ is the maximum of u, $w_x = u_{xx} < 0$ and $f'(0) < 0$. By rescaling $f \to k^3 f$ and $\xi \to k\xi$ in the equation (36), one can always achieve $f'(0) = -1$.

We show now that the asymptotics of equation (36) when $\xi \to \infty$ is given by

$$f(\xi) \simeq C\xi^{-\alpha/\beta} + \phi(\xi), \tag{37}$$

where $\phi(\xi)$ is small compared to $C\xi^{-\alpha/\beta}$ in this limit. For the physically meaningful values $\alpha \geq 0$ and $\beta < 0$, $-\alpha/\beta \geq 0$ and, since $f(\xi)$ is supposed to be negative for $\xi > 0$, we have $C < 0$.

The power law contribution $C\xi^{-\alpha/\beta}$ in (37) cancels the two terms on the right hand side of (36) to leading order. Since ϕ is small, the non-linear term f^2 in (36) can be linearised $f^2 = C^2\xi^{-2\alpha/\beta} + 2C\xi^{-\alpha/\beta}\phi(\xi)$, as the term of the order ϕ^2 can be neglected. Of the remaining terms, the highest derivatives of $\phi(\xi)$ turn out to be dominant, because $\phi(\xi)$ is a rapidly oscillating function (see below). One can see that it was necessary to compute the sub-dominant asymptotic ϕ in the expression (37), since the terms containing ϕ become dominant upon the substitution of (37) into (36). The equation (36) then simplifies to

$$-\beta\xi\phi' = 2C\xi^{-\alpha/\beta}\phi'''. \tag{38}$$

The asymptotics of (38) can be obtained by the WKB method. We try

$$\phi'(\xi) = e^{S(\xi)}. \tag{39}$$

Then, (38) transforms to

$$S'^2 + S'' = -\frac{\beta}{2C}\xi^p, \tag{40}$$

where

$$p = 1 + \alpha/\beta = -2 - 1/\beta. \tag{41}$$

One can see that $p > -2$ since $\beta < 0$. In this case, $S'^2 \gg S''$ in (40) and we get the asymptotic relation

$$S(\xi) \sim i\sqrt{\frac{\beta}{2C}}\frac{1}{1+p/2}\xi^{1+p/2} - \frac{p}{4}\log\xi + a + \dots, \tag{42}$$

Here a is an arbitrary constant. Since $\phi'(\xi) = e^{S(\xi)}$, (37) transforms to

$$\phi'(\xi) \sim A\xi^{-p/4}\sin(\omega\xi^{1+p/2}). \tag{43}$$

where p is given by (41), $A = e^a$ is an arbitrary constant and the "frequency" ω is given by

$$\omega = \frac{2}{2+p}\sqrt{\frac{\beta}{2C}}. \tag{44}$$

Since $\beta < 0$ and $C < 0$, the "frequency" ω in (44) is real. Thus, the solution f for equation (36) is asymptotically given by

$$f(\xi) \simeq C\xi^{-\alpha/\beta} + A_1\xi^{-3p/4}\sin(\omega\xi^{1+p/2}), \tag{45}$$

where $A_1 = A(1+p/2)^{-1}$.

To compare the asymptotic expression (45) with the exact solution of the equation (36), we plot $\log(|f(x)|)$ versus $\log(x)$, for $\alpha > 0$ on figure 3 and figure 4. The slope of the straight line is $-\alpha/\beta$. We see that the solution rapidly approaches the asymptotic form $f(\xi) \sim C\xi^{-\alpha/\beta}$.

Only the choice $\alpha = 0$ can satisfy the condition that f tends to a constant at infinity. In this particular case, $p = 1 + \alpha/\beta = 1$ and the asymptotic expression (45) becomes

$$f(\xi) \simeq C + A\xi^{-3/4}\sin(\omega\xi^{3/2}). \tag{46}$$

Note that in this case the solution of equation (38) can be expressed in terms of Airy functions. In fact, equation (38) becomes

$$-\beta\xi\phi' = 2C\phi''', \tag{47}$$

which is just a version of the Airy equation

$$y'' - zy = 0 \tag{48}$$

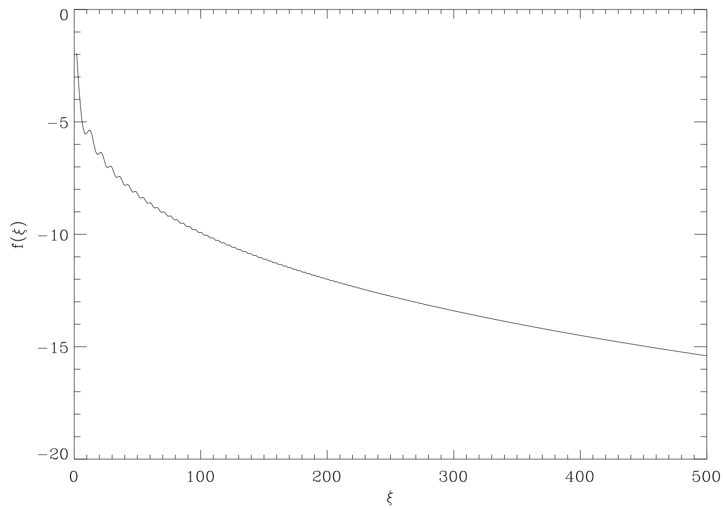

Figure 3: The self-similar solution with $\alpha = 0.1$.

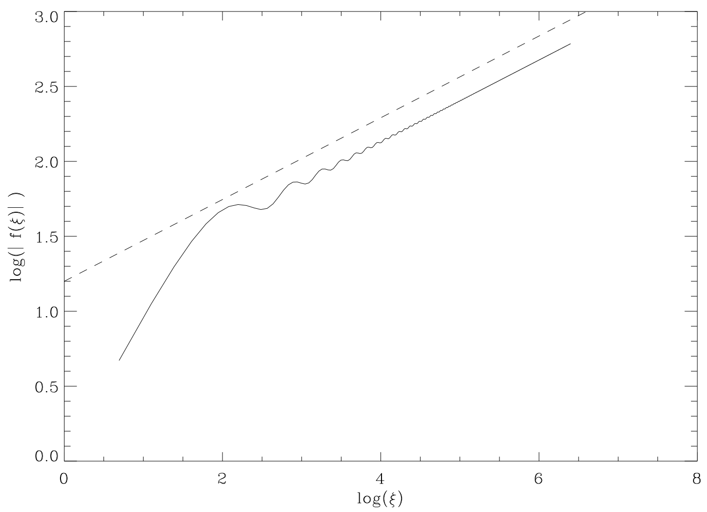

Figure 4: Double logarithmic plot of the self-similar solution with $\alpha = 0.1$ (figure 3) (solid line) showing the asymptotics $f \sim \xi^{-\beta/\alpha}$ (dashed line).

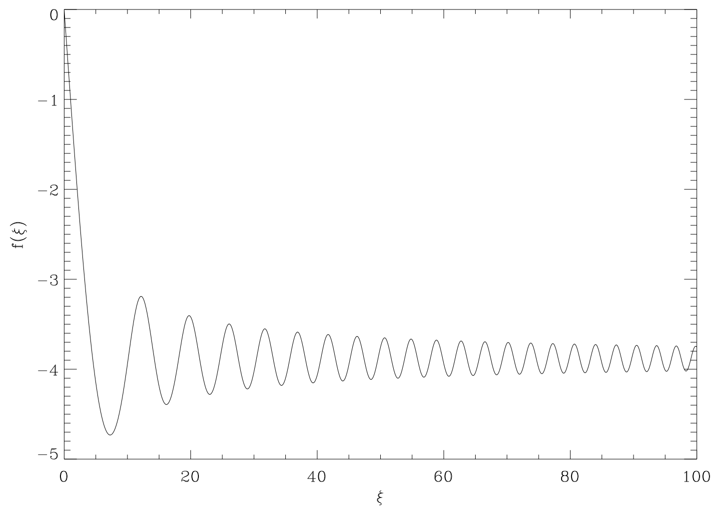

Figure 5: The self-similar solution approaching a constant for $\alpha = 0$

for $y = \phi'(\xi)$. One has to change sign and rescale the variable ξ to obtain the standard form of the Airy equation for $\phi'(\xi)$ from (47). The asymptotics for the Airy equation (48) when $z \to -\infty$ is

$$y(z) \simeq |z|^{-1/4} \sin(2/3|z|^{3/2} + \theta), \tag{49}$$

which is identical with the expression (43) with $p = 1$. Here θ is an arbitrary constant.

In figure 5, we plot the solution of (36) with $\alpha = 0$. One can see the solution approaches a constant when ξ tends to infinity.

To check the law of the oscillations of subdominant term in (46), we plot the position of the k-th extremum of the solution to (36) versus k on figure 6. If (46) is correct, we have $\xi_k^{3/2} \sim k\pi/\omega$ for large k. The slope of the straight line is 2/3. We see that the asymptotic prediction (46) works very well.

We also check the decay of the amplitude of the subdominant term $\phi(\xi)$ in (46). We compute f_k, the value of the function f at the k-th extremum, and consider the difference $|f_k - f_{k+1}|$. According to (46), this difference must decay as $\xi^{-3/4}$. We present the results of our computations on figure 7. The slope of the straight line is $-3/4$, and again the theoretical prediction (46) works well.

4.2 Stability of the self-similar solution

To investigate the stability of the self-similar solution (35) with $\alpha = 0$ and $\beta = -1/3$, we have made a simulation of the "inner part" of the singularity. As t approaches t_0, the ansatz (35) will become valid in an x-domain which shrinks to 0. Substituting

$$t \to t, \xi = (t_0 - t)^{-1/3} x \tag{50}$$

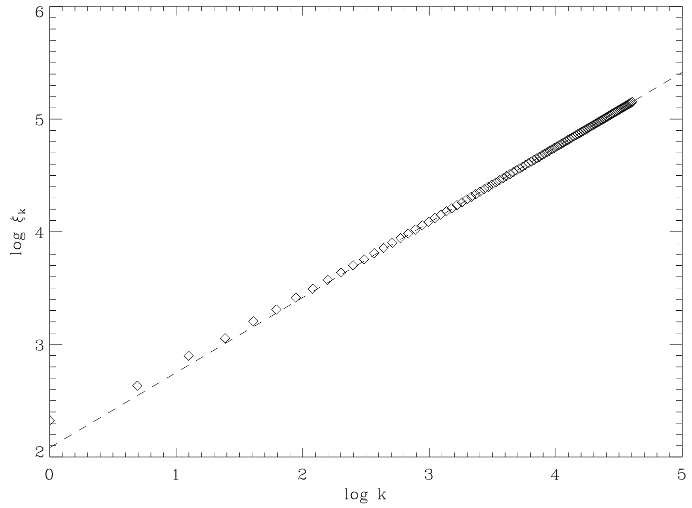

Figure 6: The positions of extrema of the self-similar solution with $\alpha = 0$ (figure 5) compared to the asymptotics (46) (shown as a dashed line), i.e. $\xi_k \sim (\omega k/\pi)^{2/3}$ with ω given by (44).

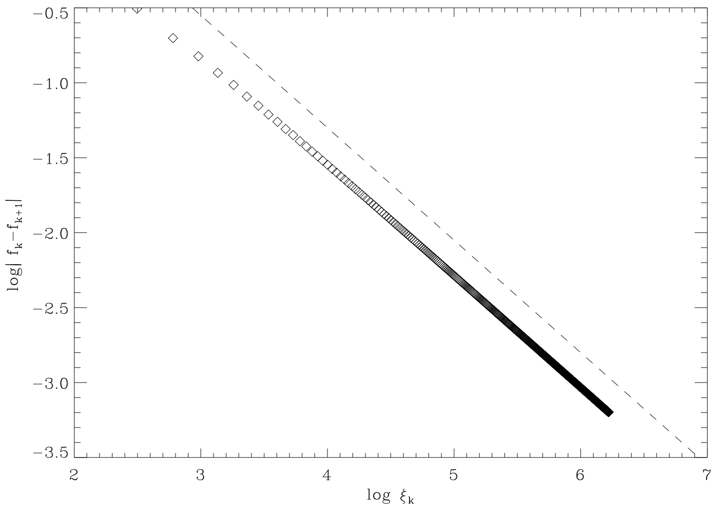

Figure 7: The value $|f_k - f_{k+1}|$ (proportional to the amplitude of the oscillations of $\phi(\xi)$) versus the position of the k-th extremum ξ_k (diamonds) compared to the asymptotics $\xi^{-3/4}$ (dashed line).

into (10), we get

$$(t_0 - t)w_t + \frac{1}{3}\xi w_\xi = (w^2)_{\xi\xi\xi}. \tag{51}$$

We can make an additional substitution $\tau = -\log(t_0 - t)$, so that $\tau \to +\infty$ when $t \nearrow t_0$. This substitution gets rid of the explicit t dependence in (51):

$$w_\tau + \frac{1}{3}\xi w_\xi = (w^2)_{\xi\xi\xi}. \tag{52}$$

Assume now that $W(\xi)$ is a stationary solution of the equation (52), and denote the deviation from this stationary solution by $s(\tau, \xi)$, that is $s(\tau, \xi) = w(\tau, \xi) - W(\xi)$. The equation for $s(\tau, \xi)$ is then

$$s_\tau + \frac{1}{3}\xi s_\xi = (2sW + s^2)_{\xi\xi\xi}, \tag{53}$$

where the numerically determined scaling function $W(\xi)$ is used. If the solution $s(\tau, \xi)$ of the equation (53) converges to the fixpoint $s = 0$ when $\tau \to \infty$ (which corresponds to $t \nearrow t_0$), then the approach to the singularity has a self-similar form (35) with f given by (36). To test the nonlinear stability of this fixpoint, we have solved equation (53) numerically in the interval $0 \le \xi \le M$.

Here $s(\xi, t)$ is an odd function of ξ, hence $s(0, t) = s_{\xi\xi}(0, t) = 0$. We used the "free" boundary conditions for s at $\xi = M$, i.e $s_\xi(\tau, \xi = M) = 0$. For the discretization close to $\xi = 0$, one had to use the points with $\xi < 0$, i.e., with $i < 0$. The fact that the solution s is odd was then explicitly used by putting $s_{-i} = -s_i$.

In our numerical simulations, we always find that s decays to 0. Thus, there is strong numerical evidence that the self-similar solution (35) with $\alpha = 0$, $\beta = -1/3$ is nonlinearly stable, that is, it is chosen by the dynamics at least for initial conditions lying in some neighbourhood of the ansatz (35).

When $t \nearrow t_0$, the stationary solution of (53) has to be scaled by $(t_0 - t)^\beta$. Hence, the form of w close to the point $x = 0$ becomes more and more like a step function with the amplitude C. The self-similar solution has to be matched to the "outer" solution of (10), which will have the inner limit $w(x \to 0, t) = \pm C$. When $t = t_0$, the self-similar solution vanishes and only a step-function is left.

4.3 Decay of shocks

From figure 1 one can easily observe that the shocks decay in time. We shall now show that the decaying shocks have a self-similar form and their decay can be computed analytically.

We seek the decay of the shock in the form, where t and x variables are separated:

$$w(x, t) = T(t)F(x). \tag{54}$$

Substituting this expression into (10), we get two equations for the functions $T(t)$ and $F(x)$:

$$\frac{\dot{T}}{T^2} = \frac{(F^2)'''}{F} = \lambda. \tag{55}$$

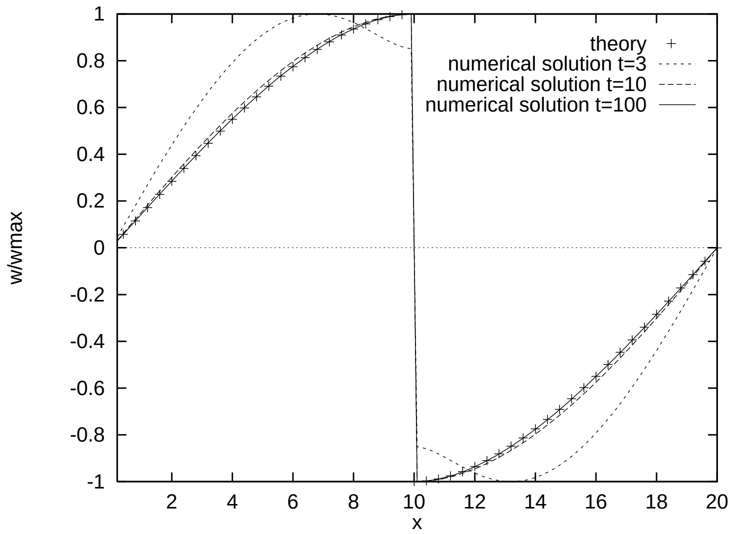

Figure 8: Snapshots of the solution $w(x,t)/w_{max}(t)$ (full line) versus the solution of (55) (crosses). The dotted line shows the initial condition. Periodic boundary conditions, $L = 20$.

The solution for the time function $T(t)$ is then

$$\frac{1}{T} = \frac{1}{T_0} - \lambda t. \tag{56}$$

We see that, provided $\lambda < 0$, T decays approximately as $1/t$ for large t.

The eigenvalue λ depends on the number of shocks we want to describe. If we chose the initial conditions to be $w_0(x) = \sin(2\pi x/L)$ for $0 \leq x \leq L$, there will be only one shock at $x = L/2$ and the solution $w(x,t)$ remains antisymmetric around $x = 0$: $w(-x,t) = w(L-x,t) = -w(x,t)$. Here we have assumed the periodic boundary conditions.

We assume that the maximum of F is equal to 1, (i.e $F(x = L/2 - 0) = 1$), since the amplitude can be absorbed into the function T. $F(x)$ is discontinuous at $x = L/2$. From figure 1 we observe that, for large t, the slope of w at the shock is zero, therefore, we require $F'(x = L/2 - 0) = 0$. The function F is assumed to be an antisymmetric function of x around $x = 0$, so that $F(0) = F''(0) = 0$. This specifies the eigenvalue problem for the function F. Solving it numerically, we obtain $\lambda \simeq -0.01887$ for $L = 20$. In figure 8 we plot the rescaled solution $w(x,t)/w_{max}(t)$ and compare it with the solution of the eigenvalue problem $F_\lambda(x)$. Snapshots of the numerical solution for different times collapse onto one line which agrees very well with the theoretical prediction, i.e the function $F_\lambda(x)$.

In figure 9 we plot $1/w_{max}(t)$ versus t and compare it with the predicted straight line (56). Again, the agreement is very good.

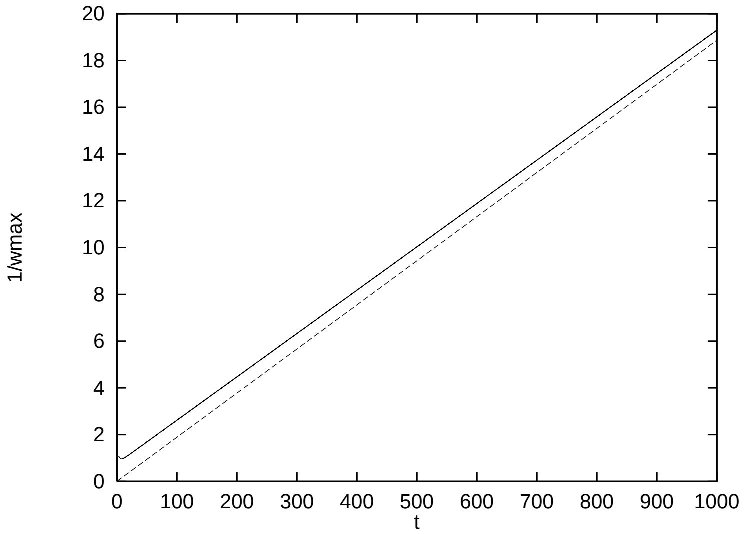

Figure 9: The decay of a shock. The full line shows $1/w_{max}(t)$ and the dashed line has the slope $-\lambda$, predicted by (56).

5 Numerical simulations

5.1 Explicit method

We have managed to construct an explicit scheme for simulating equations (1) and (9) with arbitrary values of the viscosity, even with $\nu = 0$. This scheme is robust and simple, and we shall strongly recommend it for use in the extensive numerical simulations like the ones performed in [7].

In what follows, we shall work with the differentiated equation (9) since it is more illustrative. The analogue of the code for the equation (1) can then be easily constructed.

We believe that the instability reported in [7] arises from the discretization of the non linear term w^2_{xxx}. In fact, one can chose either left or right discretization at the point $x = x_i$:

$$\begin{aligned} w^2_{xxx,l} &= \tfrac{1}{h^3}(w^2_{i+1} - 3w^2_i + 3w^2_{i-1} - w^2_{i-2}) \\ w^2_{xxx,r} &= \tfrac{1}{h^3}(w^2_{i+2} - 3w^2_{i+1} + 3w^2_i - w^2_{i-1}). \end{aligned} \tag{57}$$

Here h is the lattice spacing and $w_i = w(x_i)$. The trick is to chose the discretization $w^2_{xxx,l}$ at the point x_i if $w(x_i) > 0$ and, correspondingly, $w^2_{xxx,r}$ if $w(x_i) < 0$. The discretization of the viscous term νw_{xxxx} is chosen to be symmetric around the point $x = x_i$. With such a discretization chosen, no problems are encountered in simulating the equation (9) with arbitrary values of ν, even $\nu = 0$. This is reminiscent of the "up-winding" methods used to simulate hyperbolic quasilinear equations [17]. The scheme is stable provided that the time step is small enough, the condition for stability is

$$\tau(2 \max|w|h + \nu) < h^4, \tag{58}$$

where τ is the time step, and h is the distance between the points in the spatial grid. The analogous "center" scheme $w^2_{xxx} = (w^2_{xxx,l} + w^2_{xxx,r})/2$ or one-sided discretization ($w^2_{xxx} = w^2_{xxx,l}$ or $w^2_{xxx} = w^2_{xxx,r}$) have the same stability condition (58). As long as the solution is smooth, the schemes are equivalent, however only the "up-wind" scheme seems to describe the shock without problems. The "center" scheme could predict the shock for antisymmetric initial conditions for some time, but diverged eventually, however small the time step was. The asymmetric scheme diverged right away because it introduced asymmetric perturbations around the shock.

The results of our numerical simulation are presented in figure 1 and figure 2. For $\nu = 0$, we clearly see the formation of shocks at the maxima of u, i.e at the places where w changes sign from $+$ to $-$ (moving from left to right). This is precisely the kind of singularity investigated in Section 4. The shock then slowly decays as discussed above, but does not seem to move, which indicates that w^2_{xx} is continuous across the shock. The simulations can be continued for an arbitrarily long time, as long as the stability criterion (58) remains satisfied.

For $\nu > 0$, all the solutions behave in a similar manner: first, the maximum of u sharpens, although shocks are never quite formed. Finally, the solution slowly decays.

5.2 Implicit method

A second set of simulations was performed to attempt to show that the real solution to the equation (10) is in fact approaching the singularity in the self-similar way described above.

When a singularity is formed, one would like to have a finer mesh close to it. However, from the stability criteria of the explicit method (58) we see that mesh refining will require very small time steps. Thus, if one wants to resolve the region close to the singularity, one has to use implicit methods, since they are stable for large time steps.

Suppose we know the solution $W(x)$ of equation (10) at time t. Now we attempt to find the solution $w(x)$ at time $t + \tau$, where τ is the time step. The implicit Euler method gives:

$$\frac{w - W}{\tau} = (w^2)_{xxx}. \tag{59}$$

We discretize it in the following way:

$$\frac{w_i - W_i}{\tau} = \frac{1}{h^3}(w^2_{i+2} - 3w^2_{i+1} + 3w^2_i - w^2_{i-1}) \tag{60}$$

Here we have assumed that the distance between the grid points in x is constant, the reader can easily generalize this to the case when the spatial grid is not equally-spaced. The boundary conditions are assumed periodic, and the solution is antisymmetric around $x = 0$, i.e., $w(x,t) = -w(-x,t)$. The stability analysis shows that this scheme is absolutely stable. However, one has to solve the set (60) of non-linear equations in every time step. This is done by minimizing the error functional for the system (60), which we define to be

$$Err(w) = \sum_i \left(\frac{w_i - W_i}{\tau} - \frac{1}{h^3}(w^2_{i+2} - 3w^2_{i+1} + 3w^2_i - w^2_{i-1}) \right)^2.$$

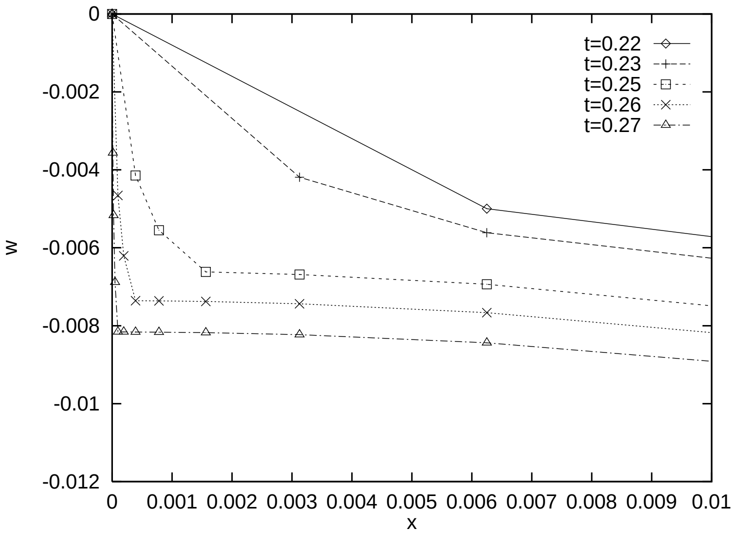

Figure 10: Numerical solution of the inviscid w equation (10) by the implicit method, periodic boundary conditions. Only the part $x > 0$ is plotted, since the solution is presumed to be antisymmetric around the point $x = 0$. It is seen that a shock is formed at $x = 0$.

The functional $Err(w)$ can be minimized using standard methods. We have used the subroutine E04FDF from the software package NAG. Using the old solution W as initial approximation to the position of the minimum, this method works satisfactorily when the number of points is not very large. We start with an equally-spaced grid and add points in the regions of large gradients (the local density of points is kept proportional to the local gradient of the solution). The results of a numerical simulation using this method is shown in figure 10.

We start with initial condition $w(x, t = 0) = -\sin(2\pi x/L)$. Quite quickly, a small singularity appears near $x = 0$. The singularity is so small that it is almost invisible if the solution in the whole x interval is plotted. That is why we chose to plot the solution close to $x = 0$.

In figure 11, we plot the value of $1/w_x(x = 0)$ versus time (this quantity describes the "width" of the singularity). From our previous discussion it follows that $1/w_x(x = 0) \sim (t_0 - t)^{-\beta} = (t_0 - t)^{1/3}$. The numerical solution does suggest that the width becomes zero at finite time, but we cannot resolve the form of the singularity.

Finally, in figure 12, we plot $w(x)$ versus $\xi = xw_x(x = 0)$. The solution, plotted in these coordinates, should collapse onto the graph of self-similar function, as described in Section 4.1. We see, however, that the collapse is not so good, especially for large values of ξ. We estimate that for the chosen initial conditions the singularity is formed at $t_* \simeq 0.27$.

We would like to emphasize that our numerical simulation of the shock formation constitutes at best suggestive evidence that the singularity is in fact formed as described in Section 4. Clearly, more work needs to be done in this case.

Note that one could also linearize the right-hand side in (59), for example,

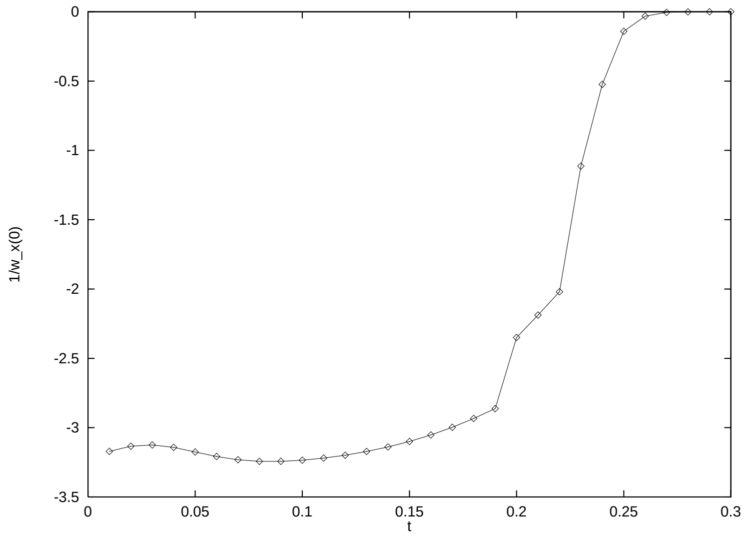

Figure 11: The "width" of the singularity $1/w_x(x=0)$ versus time.

in the following way:

$$\frac{w - W}{\tau} = (wW)_{xxx}. \tag{61}$$

The stability analysis shows that this scheme is unstable to certain wavenumers of the order $kh \sim \pi$ irrespective of the value of time step, at least if one choses the discretization similar to (60) or schemes discussed in the previous chapter. The instability of the scheme shows first around the point $w = 0$, which makes it inadequate for our purposes.

6 Conclusions

In this paper we have provided global bounds for the solutions of equations (1) and (8). Apart from its intrinsic interest, we regard this work as a necessary prerequisite for the studies of the *noisy* equations relevant in the context of surface dynamics [1, 2, 3, 4] – clearly the presence of finite time singularities in the deterministic equations would have dramatic consequences for the noisy problem [7].

We noted in the Introduction that in the physical realizations of equations (1) and (8) the nonlinearity would typically appear with a negative sign, leading to a sharpening and, in the inviscid case, to the appearance of singularities at the *minima* of the height function. Indeed, the formation of deep, almost singular grooves is observed in many simulations of discrete stochastic growth models believed to be governed by equation (1) on large scales [3, 18, 19, 20]. In the presence of noise these quasi-singularities seem to appear and disappear in a random manner, which may be related to the 'intermittent' scaling properties of certain of these models [6, 18, 20].

We have been able to prove that the solutions of (8) remain bounded by finding several new integrals of motion. This result holds for all initial condi-

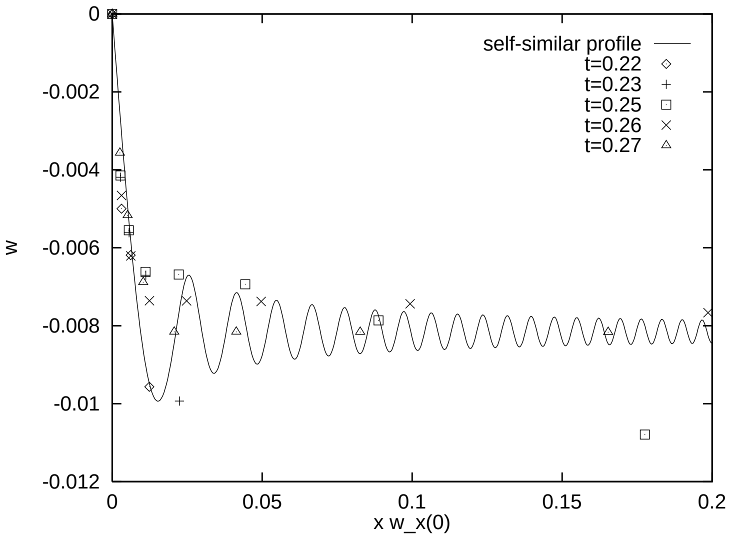

Figure 12: Numerical solution of the inviscid w close to the singularity versus $\xi = xw_x(x=0)$. Also,the self-similar function f is plotted.

tions. We present a complete picture of the evolution of generic initial data. The maxima of u will, for $\nu = 0$, inevitably lead to shocks in which u_x changes sign. We emphasise that the numerical method presented allows us to continue our simulations through the formation of shocks and observe their decay, which is well-described by a separated variables solution.

For the complete, viscous equation (1) we established exponential decay of the L_2-norm of the solution. Our proof of boundedness in the C-norm for $\nu > 0$ can be carried out only in the case when the initial conditions are very small (in this case, we also establish that the C-norm of the solution goes to zero exponentially). The presence of a finite ν smoothens the shocks and no singularities occur.

Finally, we present a numerical method, which allows us to treat arbitrary initial conditions and arbitrary viscosities $\nu \geq 0$. Thus our method should finally make it possible to simulate this equation in the presence of noise for arbitrarily long times, which is important in applications to surface growth.

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A The Poincaré Lemma and the Sobolev Embedding Theorems

If we have a function defined on the interval $[0, L]$, then the maximal value of the function is dependent on the maximum of the derivative of this function. More precisely, the following statement holds [21]:

$$sup_{x\in[0,L]}|f(x)| \leq \sqrt{L}\left(\int (f'(x))^2 \mathrm{d}x\right)^{1/2} \tag{62}$$

for all functions f such that:

1. The integral on the right-hand side of the equation (62) is finite.
2. $f = const$ obviously does not satisfy the inequality (62) and must somehow be ruled out. It will be required, therefore, that either $f(0) = f(1) = 0$ (rigid boundary conditions) or

$$\int f(x)\mathrm{d}x = 0$$

for the periodic boundary conditions.

The inequality (62) will be referred to as the Poincaré lemma. We shall note that this estimate is sharp.

In a similar way, the following inequality holds given the assumptions on the function f stated before:

$$\int (f(x))^2 \mathrm{d}x \leq L^{-2}\int (f'(x))^2 \mathrm{d}x, \tag{63}$$

which is called Poincaré inequality.

A more general result can be formulated. Let D be an open bounded domain in the n-dimensional space. Consider the Sobolev space W_p^l: The space of functions with the (generalised) derivatives up to lth order in $L_p(D)$. We can define the norm

$$\|f\|_{W_p^l} = \sum_{k=0}^{l}\left(\int |f^{(k)}(x)|^p \mathrm{d}x\right)^{1/p}. \tag{64}$$

Let $f(x) \in W_p^l$. Then, if $pl > n$, for some $K > 0$

$$\|f\|_C(D) = sup_x|f(x)| \leq K\|f\|_{W_p^l} = \sum_{k=0}^{l}\left(\int |f^{(k)}(x)|^p \mathrm{d}x\right)^{1/p}. \tag{65}$$

Here $f^{(l)}$ is the generalised l-th derivative of the function f. The inequality (65) is called the Sobolev embedding theorem (see [21]).

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