A Survey of Spiral Wave Studies: Dynamics and Symmetries

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(Dated: June 15, 2012)

We consider the dynamics of spiral waves and some of the group theoretical methods that have been used to study the nonlinear dynamics of such excitations. Namely, we survey the main results of Barkley [1] and Biktashev et al. [2] which respectively focus on spiral wave meander and group-orbit symmetry reduction.

1. INTRODUCTION

In the simplest terms, spiral waves are rotating waves which travel in stationary media. They are ubiquitous and equally as important as they are picturesque, playing a key part in physical processes such as the onset of fibrillation in the heart [3, 4]. In such a case, as explained in Ref. [3], the spontaneous breakup of an isolated spiral of electrical activity in the heart muscle leads to the creation of numerous other spirals which settle into a state known as a 'spiral glass'. This state, filled with complex pathways, prohibits the rhythmic pumping of the heart muscle and leads to sudden death. One feature ever present during spiral wave breakup is the "meander" of the wave's tip. Meandering is a term used to describe the complex trajectories followed by the tip, in contrast to the circles traced out when the wave is rigidly rotating.

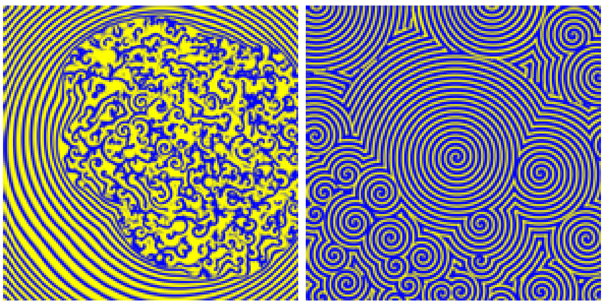

FIG. 1: Spiral wave breakup as a (possible) consequence of a Hopf bifurcation. Adapted from Ref. [5].

In order to understand these complex meandering dynamics, Dwight Barkley [1] performed a detailed bifurcation analysis of spiral waves using the reaction-diffusion equations and found that the dynamics were centered around a resonant Hopf bifurcation where eigenmodes of the bifurcation interacted with those of the system's Euclidean symmetry. Using symmetry arguments, Barkley was ultimately able to propose a low-dimensional nonlinear ODE model which efficiently captures the meandering dynamics. Later, the underlying symmetry of the problem was once again exploited by Biktashev et al. [2] in order to expose a general group-theoretic method in which the notion of group orbits and quotient systems play a key role. In the latter method, one wishes to obtain a 'generic' system (one without symmetry) by separating orbits in the full state space into a superposition of those along the relevant symmetry group; in this case $E(2)$.

In this exposition, we will discuss the main results given by Barkley and Biktashev et al. in their studies of spiral wave dynamics.

2. REACTION-DIFFUSION EQUATIONS AND THE EUCLIDEAN GROUP E(2)

Most commonly, the model used to study spiral waves consists of the semi-linear parabolic reaction-diffusion equations:

$$\partial_t u = \mathbf{D}\nabla^2 u + f(u) \tag{1}$$

where $u(\mathbf{r}, t) = (u_1, u_2)$ and $\mathbf{r} = (x, y) \in \mathbb{R}^2$. The components of u may represent the concentration of a substance (often chemical) and $\mathbf{D}$ is a matrix of diffusion coefficients. This system of PDEs is invariant under time shifts and under all isometries of the plane, the latter forming the non-compact Lie group $E(2)$ (the Euclidean group in $\mathbb{R}^2$) under composition. In general, $E(n)$ is the group of all isometries in $\mathbb{R}^n$. This group is a semidirect product of the groups $O(n)$ (reflections, rotations) and $T(n)$ (translations), i.e. $E(n) = T(n) \rtimes O(n)$. This means that for $g \in E(n)\ \exists\, o \in O(n)$ and $t \in T(n)$ such that $gx = ox + t\ \forall\, x \in \mathbb{R}^n$.

The system in (1) is *equivariant* under $E(2)$ which, following the notation in Ref. [2], means that if $u(\mathbf{r}, t)$ is a solution then given the action $T(g)$ of $g \in E(2)$, $\tilde{u}(\mathbf{r}, t) = T(g)u(\mathbf{r}, t)$ is also a solution. The action of g on u is formally defined as $T(g)u(\mathbf{r}, t) = u(g^{-1}\mathbf{r}, t)$. The isotropy subgroup of the solutions is trivial $E(2)_x = \{e\}$ with e the identity; that is to say that no solution is invariant under isometries other than the identity [2]. The spiral waves examined in these two papers are of this form.

It is the equivariance of spiral waves under this continuous symmetry group that will play a key role in studying complex meander. For a detailed, mathematical, and more general discussion of $E(n)$-equivariant PDEs, see Ref. [6].

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3. BARKLEY

In this section, we will present the results of Ref. [1].

3.1. Bifurcation Analysis

Barkley starts by studying the dynamics of a single isolated spiral wave through the equations:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \nabla^2 u + \epsilon^{-1} u(1-u)\left[u - \frac{v+b}{a}\right], \\ \frac{\partial v}{\partial t} &= u - v\end{aligned} \quad (2)$$

with a,b, and $\epsilon \ll 1$ bifurcation parameters. The fields u and v correspond to u_1 and u_2 from (1) and v is taken to be diffusionless.

After extensive numerical study of (2), Barkley was able to construct a phase diagram for the spiral wave dynamics shown in Figure 2. In this diagram, one is clearly able to see the different parameter regions where distinct patterns of meander take place. There are three main regions qualified by: no spiral waves (N), stable rotating waves (RW) (rotating at spiral frequency ω_1), and modulated rotating waves (MRW) (rotating at spiral frequency ω_2).

The RW states are the solutions of the eigenvalue problem for u and v, their tip paths form closed circles. These states become unstable under a supercritical Hopf bifurcation whose smooth curve, separating the MRW and RW regions, is shown in Figure 2. This curve is everywhere supercritical and introduces a new spiral frequency ω_2 which produces modulated waves.

As a reminder, in a Hopf bifurcation a fixed point of the system loses its local stability when a pair of complex conjugate eigenvalues from the Jacobian cross the imaginary axis as the bifurcation parameter varies. A *supercritical* Hopf bifurcation can be thought of as a case where the decay of a disturbance has turned to growth under the change of bifurcation parameter. This will ultimately change a stable spiral into an unstable one with a small surrounding limit cycle.

Once a RW state crosses the supercritical curve, it becomes quasiperiodic and its tip forms "flowers" which do not close; these are the MRW. There is also a point of resonance where $\omega_1 = \omega_2$, denoted as the modulated traveling wave (MTW) region (dashed curve). The MTW increase their translational speed with distance from the point of resonance. It is this MTW region that separates the MRW region into two: one with inward petals ($\omega_1 > \omega_2$) and one with outer petals ($\omega_1 < \omega_2$).

Since the supercritical curve is clearly the center for dynamics, it pays to look closer at the bifurcation in order to explore meander. For a RW u, Barkley examines its five leading eigenvalues $\lambda = 0, \pm i\omega_1$, and $\pm i\omega_2$. The eigenvalues associated with the Hopf bifurcation are $\lambda_H = \pm i\omega_2$ whose $\mathrm{Re}(\lambda) > 0$ as one varies a and b causing the modulation of the wave with what he terms a meandering instability. The eigenvalue at the origin is a result of rotational symmetry with eigenmode $\tilde{u}_R = \partial_\theta u$, θ the azimuthal angle. The eigenvalues $\pm i\omega_1$ are a result of translational symmetry with eigenmodes $\tilde{u}_T = \partial_x u \pm i\partial_y u$. The latter three eigenvalues are always on the imaginary axis. All eigenmodes interact around the codimension-two point resulting in the complex dynamics we see in meandering spiral waves.

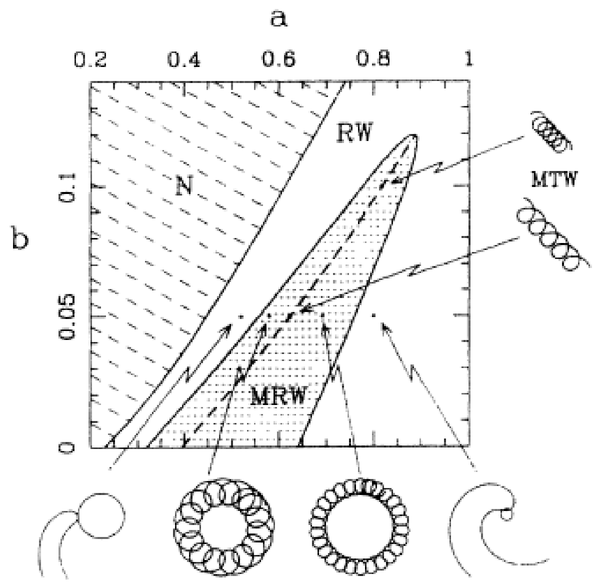

FIG. 2: Phase diagram with $\epsilon = 2 \times 10^{-2}$. The figures below and to the side display the diffrent trajectories of the spiral wave tip. The locus separating the RW and MRW regions is where the supercritical Hopf bifurcation occurs. Adapted from Ref. [1].

In order to model and further understand the meandering dynamics, Barkley proposes a weakly, low dimensional, nonlinear model which satisfies two conditions: (1) The system is equivariant under $E(2)$, and (2) it exhibits a Hopf bifurcation for RW solutions. Barkley's model is:

$$\begin{aligned}\dot{p} &= v, \\ \dot{v} &= v \cdot [f(|v|^2, w^2) + iw \cdot h(|v|^2, w^2)], \\ \dot{w} &= w \cdot g(|v|^2, w^2),\end{aligned} \quad (3)$$

with $p = x + iy$ a "position"(complex), $v = se^{i\phi}$, $s \geq 0$ velocity (complex), and w real and proportional to spiral frequency. This system is invariant under the actions of $E(2)$: translations $T_{\alpha\beta}(p, v, w) = (p + \alpha + i\beta, v, w)$, rotations $R_\gamma(p, v, w) = (e^{i\gamma}p, e^{i\gamma}v, w)$, and reflections $\kappa(p, v, w) = (p^*, v^*, -w)$. The expansions for the functions f, g, and h considered by Barkley are:

$$\begin{aligned}f(s^2, w^2) &= \alpha_0 + \alpha_1 s^2 + \alpha_2 w^2 - s^4, \\ g(s^2, w^2) &= -1 + \beta_1 s^2 - w^2, \\ h(s^2, w^2) &= \gamma_0,\end{aligned} \quad (4)$$

both α_2 and γ_0 will become bifurcation parameters and the others will be kept constant.

Further analysis by Barkley in the (s, w) subsystem reveals the full spectrum of RW, MRW, and MTW as previously observed through numerical solutions of (1). The resonance bifurcation is found to occur at values $\alpha_2 = -5$ and $\gamma_0 = \sqrt{28}$ with the full eigenspectrum as before. After normalizing these bifurcation parameters, $\mu = -(\alpha_2 + 5)/5$ and $\nu = \gamma_0/\sqrt{28}$, a new phase diagram develops which captures the complex dynamics:

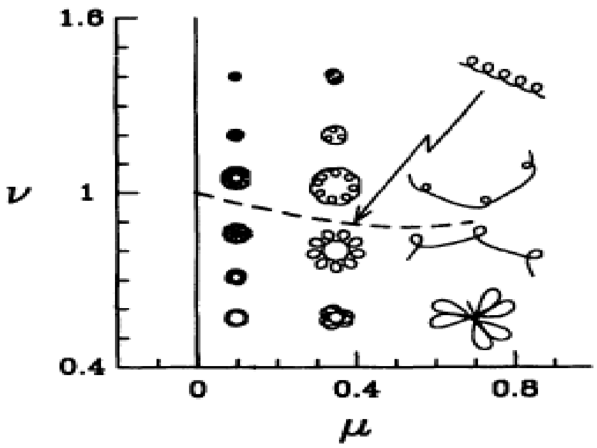

FIG. 3: Phase diagram for the Barkley model. Adapted from Ref. [1].

3.2. Conclusion

The key point in this analysis was the interaction of the Euclidean symmetry eigenmodes with the Hopf bifurcation eigenmodes as the cause of the meandering dynamics. The Barkley model given in (3) was derived on the principle of Euclidean equivariance and admittance of a Hopf bifurcation alone. According to Ref. [2], further study of this sytem shows that there is no locking between the two frequencies as a result of the $E(2)$ symmetry of the system. Studying the meander of spiral waves then becomes a problem of studying the system's Euclidean symmetry.

4. BIKTASHEV, HOLDEN, AND NIKOLAEV

In this section, we will present the results of Ref. [2].

4.1. Space Reduction Method

When a dynamical system has an underlying symmetry, we wish to remove the behavior due such structure and study the dynamics in a quotiented (or reduced) state space. Biktashev et al. note,

> It is well known that the behaviour of dynamical systems with symmetries can be drastically different from those without symmetry . . . a standard way to study symmetrical systems is to reduce them to generic ones and then apply the results of the generic theory [2]

where by 'generic' systems they refer to ones without symmetry. For dynamical system equivariant under a continuous group G, the state space is foliated by group orbits defined as $\mathcal{M}_x = \{gx \,|\, g \in G, \, x \in \mathcal{B}\}$, following the notation in Ref. [7]. The goal is to represent whole equivalence classes of group orbits by a single point in what Biktashev et al. call the orbit manifold or orbit space.

For a generic differential equation $\dot{U} = F(U)$, $U \in \mathcal{B}$ (Banach space), we wish to parameterize the state space by a manifold $\mathcal{M} \in \mathcal{B}$ which is transversal to the group orbits. A point in the orbit manifold is denoted by V and its equivalent to a point U in the full state space up to the group action $T(g)$, that is $U = T(g)V$ for $V \in \mathcal{M}$ and $g \in G$. There is also a directionality condition which only permits the group orbit to cross the orbit manifold once.

The vector field $F(V)$ for points in $\mathcal{M}$ can be decomposed in two:

$$F(V) = (F(V))_{\mathcal{M}} + (F(V))_G \tag{5}$$

The former tangent to the orbit manifold and the latter tangent to the group orbits. The sought-after quotient system is given in general form by $\dot{V} = (F(V))_{\mathcal{M}}$. This system in the manifold lacks the original symmetry and its equation of motion can later be used to recover the original space trajectory by solving for the necessary group action which maps points from the manifold to the full state space. For a geometric diagram see Figure 4.

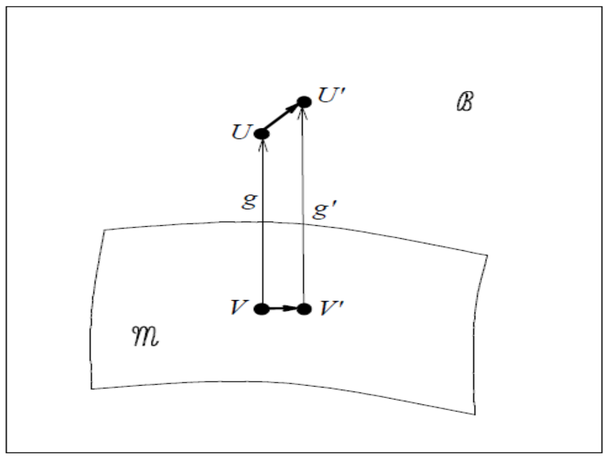

FIG. 4: Decomposing the state space $\mathcal{B}$ by use of the orbit manifold $\mathcal{M}$. Adapted from Ref. [2].

For the case of spiral waves $G = E(2)$ and $\mathcal{B}$ is no longer a Banach space but rather consists of bounded continuous vector functions, asymptotically "circular" at infinity [2]. Here, it will be shown, that separating the dynamics of the system along the orbits of $E(2)$ is visually the same as riding on the tip of the wave.

In order to choose $\mathcal{M}$ such that it satisfies the transversality condition, Biktashev et al. define a class of functions $v(x, y)$ by conditions which are violated by any motion in the plane. The transversality conditions chosen in the paper are: $v_1(0,0) = u_{10}$, $v_2(0,0) = u_{20}$, and $\partial_x v_1(0,0) = 0$. Here, the function $v(\mathbf{r})$ is just the function $u(\mathbf{r})$ moved by the action $T(g^{-1})$ along the plane. The first two choices assert an intersection point of two isolines at the origin while the last condition makes the isoline of v_1 tangent to the x-axis.

After expanding the vector field $(F(V))_G$ in the basis of the Lie algebra generators (∂_x, ∂_y, and $y\partial_x - x\partial_y$) and assuming that the full state space differential equation takes the form of (1), Biktashev et al. derive an expression for the vector field on the orbit manifold:

$$\partial_t v = \mathbf{D}\nabla^2 v + f(v) - (\mathbf{c}, \nabla)v - \omega\partial_\theta v \qquad (6)$$

This PDE, with $\mathbf{c}(t)$ a translational velocity and $\omega(t)$ a rotational velocity, is the target symmetry-free system. Namely, it is a dynamical system in the state space spanned by $\mathbf{c}$, ω, and v.

However, an equation to specify $g(t)$ is still needed. Biktashev et al. derive such an equation by formally considering the complex plane $\mathbb{C}$ and its isomorphism to $\mathbb{R}^2$. In the complex plane, $g = \{\mathbf{R}, \Theta\}$ makes a rotation by Θ and translation by $\mathbf{R}$. The action of g on $z \in \mathbb{C}$ is then defined as the mapping $T_C(g) : z \to R + ze^{i\Theta} = X + iY + ze^{i\Theta}$. The equations for Θ and R are derived in Ref. [2]:

$$\begin{aligned} \partial_t \Theta &= \omega(t), \\ \partial_t R &= c(t)e^{i\Theta} \end{aligned} \qquad (7)$$

4.2. Conclusion

What is the visual interpretation of this machinery? The transversality conditions set by Biktashev et al. move us to a frame of reference attached to the tip of the spiral. Why? Because the intersection of the isolines at the origin is the defining condition for the tip of the spiral. The function v, which is u moved by $T(g^{-1})$, is a function in this reference frame basing its origin at the tip. The third condition then makes the y-axis of this reference frame stay along the gradient of u_1 as mentioned above. This reference frame can be visualized in Figure 5 below where the coordinates (ξ, η) are the tip frame of reference coordinates. It's equation of motion is simply given by (7) while (6) gives the vector field in this frame.

In this case, the reduction to the orbit space naturally led Biktashev et al. to study the symmetry-reduced system as one with the visual interpretation of being attached to a reference frame on the spiral tip. Further numerical work based on simulating solutions to (1) and (6) is published in Ref. [8]; this publication is also equipped with the open code entitled EZRide [9] which simulates spiral wave dynamics and meander.

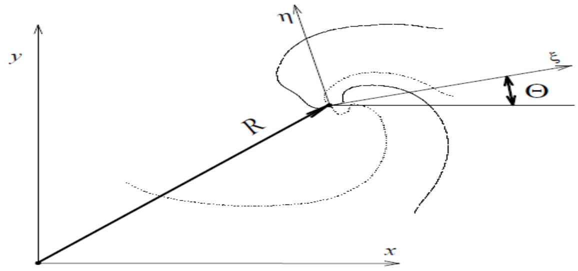

FIG. 5: The tip of the spiral is shown as the intersection of the isolines (dashed curves). The frame of reference attached to the tip has coordinates (ξ, η). Adapted from Ref. [2].

[1] D. Barkley, Phys. Rev. Lett. **72**, 164 (1994).
[2] V. N. Biktashev, A. V. Holden, and E. V. Nikolaev, Int. J. Bifur. Chaos **6**, 2433 (1996).
[3] A. Karma, Phys. Rev. Lett. **71**, 1103 (1993).
[4] A. Karma, Chaos **4**, 461 (1994).
[5] B. Sanstede, *Bjorn sanstede research group*, URL http://www.dam.brown.edu/people/sandsted/research.php?project=spirals.
[6] I. Melbourne, Trans. Amer. Math. Soc. **351**, 1575 (1999).
[7] P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner, and G. Vattay, *Chaos: Classical and Quantum* (Niels Bohr Institute, Copenhagen, 2010), ChaosBook.org.
[8] A. J. Foulkes and V. N. Biktashev, Phys. Rev. Lett. (2010).
[9] D. Barkley, V. N. Biktashev, and A. J. Foulkes, Tech. Rep., U. Liverpool (2010), URL http://www.maths.liv.ac.uk/~vadim/software/EZRide/.

Acknowledgments

I would like to gratefully acknowledge D. Barkley, V. N. Biktashev, A. V. Holden, and E. V. Nikolaev for their work presented here. I would also like to thank Dr. Predrag Cvitanović for his tireless instruction throughout the time I've known him, both in and outside the classroom.