3-disk Coulomb Pinball System

Project of Course PHYS 7123
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Introduction

Most of the systems in our world turn out to be nonlinear and chaotic, (un)fortunately. Although we can linearize many of them, often throw the baby out with the water, we lose many of the interesting nonlinear properties by doing these approximations. These nonlinear properties exist even in system with only a few degrees of freedom.

One of such interesting nonlinear systems is the 3-disk pinball system. Actually I didn't realize that the pinball game is chaotic before this course. A very smart way to study these system is to find the periodic trajectories and do cycle expansion of dynamic functions.

Model

1. Three dimensionless disks are equally spaced in a plane and their positions are fixed. Another pointlike pinball with mass m enters this region and interacts with the disks. I take the interaction between them to be attractive Coulomb force $F = -k/r^2$.

Several parameters are not fixed in this system: the mass of the pinball m, the Coulomb coefficient k and the distance between the three disks D. The Newton's equation of the billiard will be

$$m\frac{d^2\vec{r}}{dt^2} = -k\sum_{i=1,2,3}(\vec{r}-\vec{r}_i)/(\vec{r}-\vec{r}_i)^3$$

We can actually take D as our length unit and combine other parameters into the time unit, then we will have a system with no parameter at all! That means the results will be the same no matter what Coulomb system you are studying. You just need to change the time scale of your data but the topological results are the same.

2. What's the typical trajectory of this system?

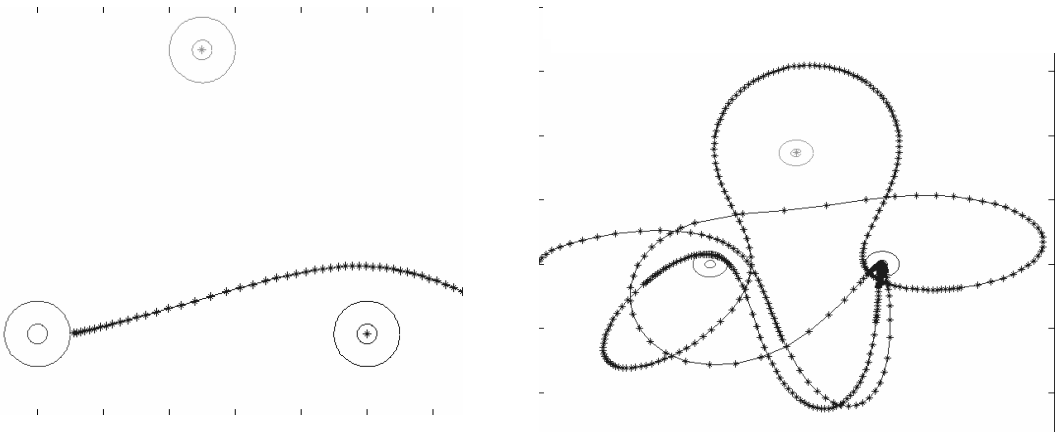


Fig. 1 Typical trajectory of the

system. The left figure is with E >0. The pinball finally escaped. The right one is E <0. The billiard stays around.

Basically we have two types of trajectories depending on the energy of the system, as shown in Fig 1. One type is with energy greater or equal than zero. That means the pinball is not confined in the system, it will escape sooner or later. The other possibilities is with energy less than zero. Then the pinball is confined. It will always move around and will never escape.

3. Why E =0?

To be chaotic system, the dynamics must be expanding and mixing. In this system, there are two ways to mix. One is from outside, which means the pinball leaves a disk and move toward the outside and because of the attractive force it will get bent and turned back to the system when the total energy is negative. Another way is that it moves close to one of the disk and deflected to another disk because of the strong interaction.

I find that it's very difficult to describe and classify the mixing from outside. So a symbolic dynamics is also almost impossible. But for the positive energy case, it's quite obvious: the pinball can change direction only if it approaches a disk close enough, otherwise it will flee away infinitely. So we can classify all the trajectories by counting what disks the pinball approaches in sequence. One another hand, system with energy greater than zero could be very interesting. A periodic trajectory will mean that a non-negative-energy pinball could actually be confined!

I guess that's one of the mechanisms that the solar system can capture some of the guests such as comets, meteors or even planets passing by.

I decided to study the case of E =0 because it's a not confined and it's the marginal point between of confined and non-confined.

Poincaré Section

To choose the Poincaré section, I defined a circle around every pinball. The circles should not be so large that they overlap. It should also not be so small that a pinball could change direction to another disk without passing that surface. In my simulation I choose the radius of Poincare's section as 0.1.

Since the energy is conserved, the only variables needed to describe the movement in the position in that Poincaré section (denoted by an angle α) and the angle between the velocity and position (denoted by φ). The Poincaré section is shown in below:

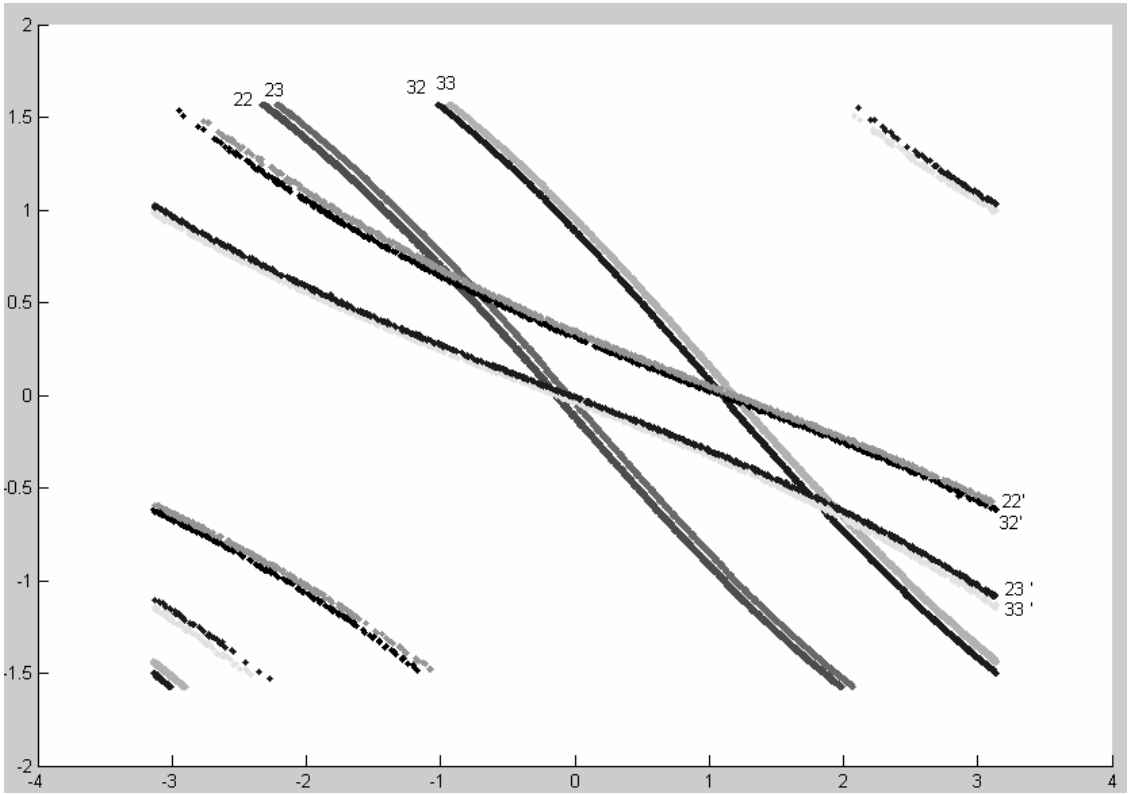


Fig. 2 Poincaré 's Section of this system. X-axis is α and y-axis is φ. Here I showed strips of 22, 23, 33 and 32. The 22 strip contains all the initial points that can make 22 mapping. The 22' strip contains all the points after the 22 mapping. So on for 23, 33 and 32 strips. Notice that those strips get wrapped across the boundary of 0 and 2π.. I also use this to guide my searching for periodic trajectories. For example, the 23 cycle could happen only at cross section of 23 and 23' strip. So I take initial condition from this region in my Newton's method.

Symbolic Dynamics

Since I already chose similar Poincaré section as the textbook. I will not object to use similar symbolic dynamics. But only a few difference for programming reasons. The pinball starts from the first disk. If it hits the second disk (the right side), the trajectory is called 2. If it hits the third one (the left side), it's trajectory is called 3. And so on for the following sequences.

So the typical cycles will be: 2, 23, 233, … ,

Please notice that the only way a pinball can change direction is by approaching a dimensionless disk. When the pinball with total energy E=0 comes very close to a disk, we can neglect forces from other two disks. And the totally energy is greater than zero with respect to a single disk, It will move in a hyperbolic trajectory. So the incoming and outgoing trajectory will never cross near a Poincaré section circle. And this trajectory can be determined only by symbols.

For example, a pinball is near disk 1. It comes from disk 2 which is on the right and it wants to go to disk 3 on the left hand. The only way is to go clockwise around disk 1 and deflects for about 120 degree. So on for other trajectories.

Cycles & Computational details:

In computation I use Runge-Kuta integration to solve the dynamic equation . The pinball starts from the Poincare section of the first disk at the left bottom. The position and velocity angle is determined by initial conditions. And speed is set so that V+T = 0, where V is the total potential energy and T is the kinetic energy of the pinball.

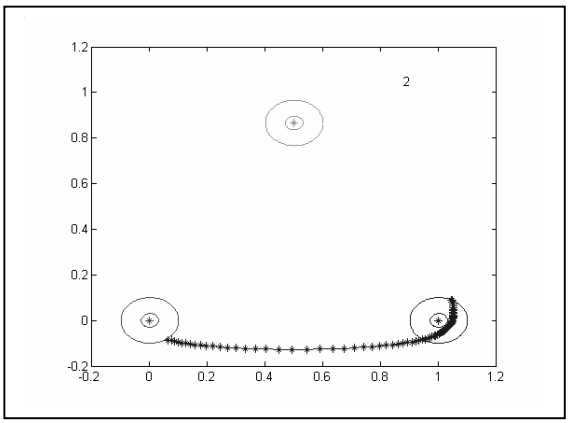
The time step is varied so that it is large in potential plain region and is small in potential deep region.

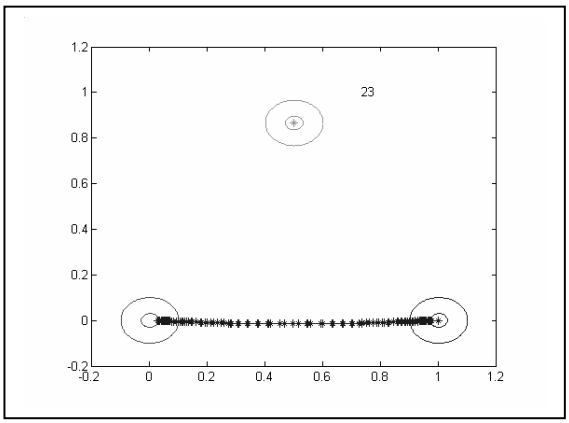
When the pinball is very close to a disk its speed will be very very high. The are two ways to deal with that. One is continue to take time step small, so that the integration is still accurate. But that will takes long time. The other way is what I use. When the pinball goes very close to a disk, here I take 0.03D, the force from other disks will be 0.1% less that the force from the nearby disk. I treat this as a standard Kepler's problem and use the formula from Goldstein's book to compute the time, position and velocity when the pinball goes out of the 0.03 circle. And I also renormalize the speed at this point to make the total energy unchanged.

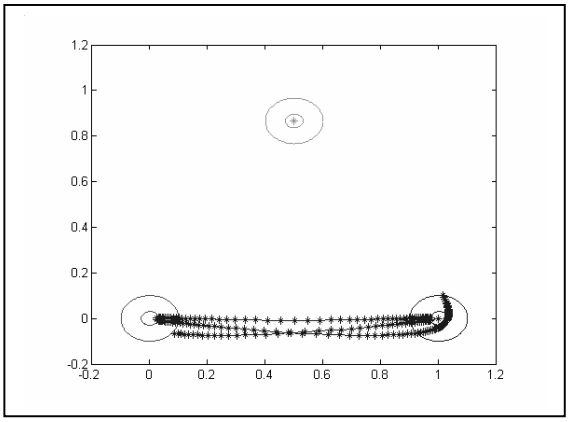
The 0.03 circle is plotted in every diagram. So the trajectory details inside the circle is invisible in the diagrams.

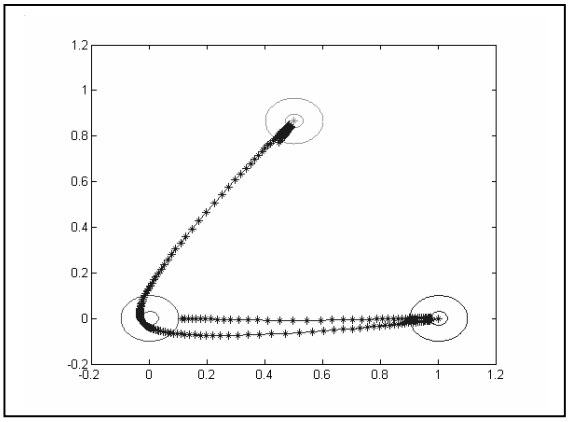
Since I only interested in the escape rate, I don't really care if the cycles really goes back to itself. I only care if it will hit another disk or escape. So What I actually do it take three Poincare section around each disk. The periodic orbits are those who go back to the initial condition in either Poincare circle.

I get the cycles using standard Newton's method. The cycles I got up to 4 bounces are listed below. The J_p for each cycle is obtained from the change on the Poincare section. And Λ from eigen values of J_p









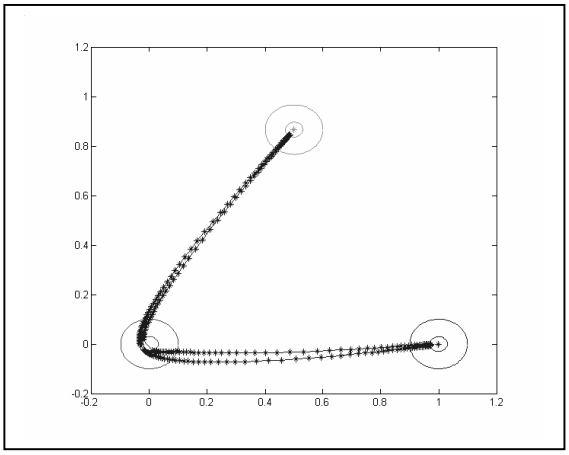


Fig 3. Cycles up to 4 bounces. Their symbols are 2, 23, 232, 233 and 2332

Dynamic zeta function:

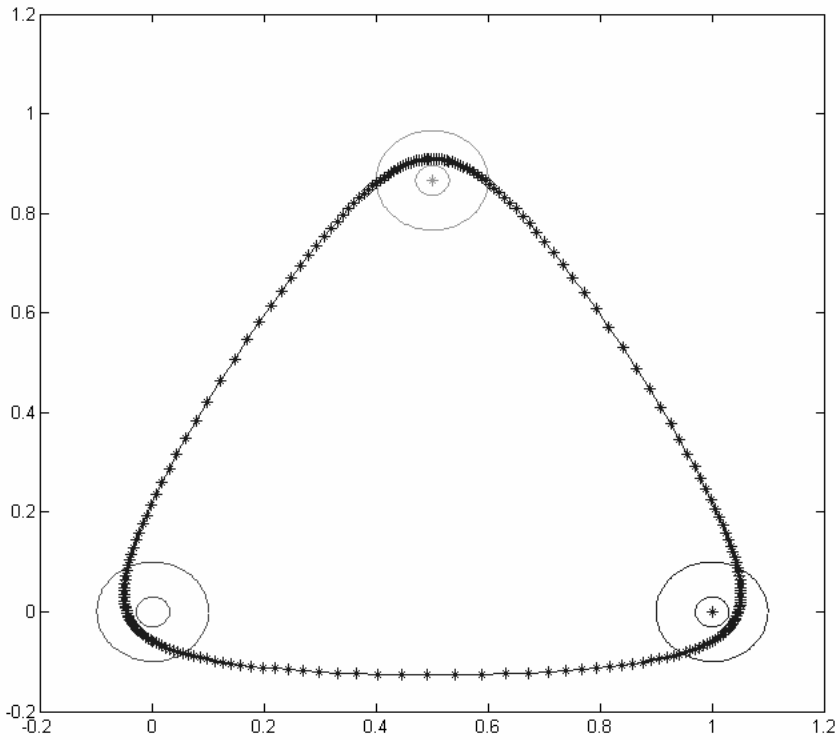
N	p (symbols from the textbook)	Λ_p	T_p	s
1	2 (NaN)	135.37	0.9297	145.6061
2	23 (0)	189.26	1.5760	
3	233 (01)	1806.06	2.4568	
3	232 (10)	1819.99	2.4227	
4	2332 (010)	20432.05	3.2739	

Table 1. Prime cycles up to 4 bounces. The cycles are degenerate because of time reversal symmetry.

$1/\zeta(s) = (1 - t_2)\,(1 - t_{23})\,(1 - t_{233})\,(1 - t_{232})\,(1 - t_{2332}) = 1 - t_2 - t_{23} + t_2^{\,2} - t_{233} - t_{232} - t_2^{\,3} + t_2 t_{23} - t_{2332} + t_2^{\,4} + t_{23}^{\,2} - t_2^{\,2} t_{23} + t_2\, t_{233} + t_2\, t_{232}$

Highlights

I find that trajectory 222 is very interesting, because it's really an orbit that moving around with energy equals zero. I believe there will also be such cyclic orbits even for energy slightly greater than zero. Notice that in most other cyclic orbits, the billiards will have to go very close to a pinball in order to get deflected to an angle big enough, which in practice will hit the pinball considering that real objects always have non-zero dimensions. But cycle 222 is different. It doesn't goes very close to any of the disks.



Drawbacks

There could be better ways to define the Poincaré 's plane. I choose the circles just to make symbolic dynamics similar to that from the textbook. But I find most of the trajectory will escape so that it will not show up in the Poincaré 's plane. That makes the task of finding cycles extremely difficult. I cannot use any more the standard methods described in Chapter 12. I spent two sleepless night the cycles and only got a few. The

zero.

most difficult thing to the keep the points and not let it disappear from my eyes. And since that, I don't have enough time to compute the zeroes of zeta function by scanning the complex s plane.

A better way may be that we include also the escaping point in the Poincaré 's plane and assign useful values so it can drive the escaping points back to normal positions. But It's Monday midnight now and I don't have time to do that.