The Diagonal Approximation and Beyond
Case study : The Spectral Two Point Function

K. Splittorff

The Niels Bohr Institute
Blegdamsvej 17
DK-2100 Copenhagen Ø
Denmark

January 20, 2000

Abstract

The derivation of the spectral two-point correlation function for classically chaotic systems in their semiclassical limit based on periodic orbit theory is reviewed. The contributions from identical pairs of periodic orbits and from non-identical pairs are evaluated. For systems without time reversal symmetry the result coincides with the exact analytic expression for the two-point correlation function of the unitary ensemble of random matrix theory. For time reversal invariant systems only the leading order for large separation is recovered.

1 Introduction

Information on the spectral properties of a system provides valuable insight into the dynamics displayed by the system. In a quantum system the spectrum of the Hamiltonian determines the resolvent which describes the propagation of particles in the system. Since the spectral information is related to the dynamics of the system it may come as a surprise when seemingly very different systems prove to have identical spectral correlators. When this occurs the correlation functions are said to show spectral universality. To elaborate on this we now define those correlation functions of the eigenvalues of an operator H which are of interest in this report

$$d(E_1, E_2, \ldots, E_k) \equiv \left\langle \prod_{i=1}^{k} \mathrm{Tr}\delta(E_i - H) \right\rangle . \tag{1.1}$$

The $\langle \ldots \rangle$ indicate an average, and is to be specified in the given context where the correlator is measured. The simplest correlator is the one-point correlation function, obtained for $k = 1$, and is also known as the spectral density. Spectral universality is the statement that the correlation functions are identical. The identity is *not* required in the full range of the spectrum. In fact, the universal correlation functions are only realized when zooming in on the correlation functions. The scale on which to look for spectral universality is set by the mean level spacing $1/d(E)$. The correlations on this scale describe the statistical properties of the oscillations around the mean correlation function, and it is exactly these correlations which often prove to be universal. We shall use the notation

$$d(E_1, E_2, \ldots, E_k) \equiv \bar{d}(E_1, E_2, \ldots, E_k) + d_{\mathrm{osc}}(E_1, E_2, \ldots, E_k) \tag{1.2}$$

to separate the average behaviour from the oscillating part.

The observations of spectral universality go back to the studies of energy levels in nuclei [1]. The challenge of describing the observed universality was taken up by Wigner and Dyson [1]. The outcome was a general theory of the statistical properties of eigenvalues. This theory is based on ensembles of matrices whose elements are drawn on fixed distributions. The energy level statistics predicted by this random matrix theory are the ensemble averaged statistics of the eigenvalues of the random matrices measured on the scale of $1/d(E)$. The extent of the applicability of the random matrix results is impressive; it has shown merit in such diverse fields of physics as atomic physics, solid state physics, and quantum chromodynamics. The theory of random matrices however only gives hints to why the correlation functions obtained are universal, and in what way the universality is realized. To answer this essential question it is necessary to go beyond random matrix theory. The theoretical challenge is to develop a theory that explains how spectral universality arises in spectral statistics. Three different theories have been examined in this context, chiral perturbation theory [2], periodic orbit theory [3], and disorder models [4]. Even though the averages involved and the technical methods are fundamentally different, the overall line of approach is identical: Rederive the results obtained in random matrix theory from the favourite theory of the field. The object of this report is to review the theoretical explanation of how spectral universality arises in spectral statistics of chaotic quantum dynamical systems.

In the field of chaotic quantum dynamics it has been realized that the spectrum of eigenvalues is dual to the spectrum of periodic orbits [5]. In the semiclassical limit, $\hbar \to 0$, this fundamental correspondence is expressed mathematically through the trace formula by Gutzwiller [6]

$$d(E) \sim \bar{d}(E) + \frac{1}{\pi\hbar}\mathrm{Re}\sum_{p}\sum_{r=1}^{\infty} A_p(E) \exp\left(\frac{i}{\hbar} r S_p(E)\right) . \tag{1.3}$$

The quantities A and S are both classical, i.e. independent of $\hbar$, A_p is a real positive amplitude of the periodic orbit p and S_p is its action. The r summation index labels repetitions of the periodic orbits labelled by p. The trace formula is only exact in the semiclassical limit. It gives the correction to the classical density to leading order in $\hbar$, this is why we have used $\sim$ and not $=$ in Eq. 1.3. (The trace formula will be discussed in further detail in section 3.)

The problem to be studied in this report can now be given a precise meaning. We will show that the average spectral correlators obtained from Gutzwiller's trace formula coincide with those derived from random matrix theory when measured in units of $1/d(E)$. To be specific we will study the two-point function $d(E_1, E_2)$. By its definition, Eq. (1.1), the two-point function is an average over two-densities $d(E_1, E_2) = \langle d(E_1)d(E_2)\rangle$. Substituting Gutzwiller's trace formula for $d(E)$ we obtain four terms, one of which includes a double sum over periodic orbits and their repetitions. The evaluation of this term is the key to show how spectral universality arises. We shall split the double sum into a diagonal part and an off-diagonal part. The two parts give the contributions from pairs of identical periodic orbits and non-identical ones respectively. While the diagonal part is evaluated through a few steps, there is at present no direct way to evaluate the contribution from the off-diagonal part. An indirect method for evaluating the off-diagonal part is given by Bogomolny and Keating [3]. It is the main assignment of this report to present and discuss this indirect method.

In order to obtain a basis for the discussion of the indirect method of Bogomolny and Keating we need to recall a couple of standard theoretical considerations. In order of appearance we present the Gutzwiller trace formula with special attention drawn to the scales involved, the principle of uniformity and the Hannay-Ozorio de Almeria sumrule which follows from this principle. Using this we find the contribution to the two-point spectral correlation function from the diagonal terms. This enables us to follow and discuss the indirect method of Bogomolny and Keating. Prior to this, we start off with an introduction to random matrix theory and universality within random matrix theory.

2 Introduction to Random Matrix Theory

Even today, four decades after the pioneering work of Dyson and Wigner [1], the problem of level statistics in quantum systems is attracting considerable interest. The reason for this persistence is simple: An ever extending class of physical systems proves to be described by the random matrix statistics of Dyson and Wigner (see e.g. [8]). As a consequence of the success, the theory of random matrices has been intensively investigated and new aspects have been proven. Most importantly is the concept of consistency[1] of the random matrix predictions within the theory itself. This consistency gives valuable hints to which of the predictions from random matrix theory we can expect to find in physical systems. In this section we give an introduction to random matrix theory. The consistency requirements, set by the theory, will guide us to look at the spectral correlation functions on the scale of $1/d(E)$. We will give the exact analytical predictions for the spectral two-point function, since this is the result which we shall derive from the theory of periodic orbits later on.

The predictions of random matrix theory come in three versions. The versions are designed to mimic the possible spectral symmetries due to invariance properties of physical systems. The names labelling the versions are the Orthogonal Ensemble (OE), the Unitary Ensemble (UE), and the Symplectic Ensemble (SE) [9]. Their properties can be summed up as follows:

[1]This consistency is referred to as universality by the random matrix theory community. But to avoid confusing this universality within random matrix theory with the universality between the spectral correlation functions of different physical systems we stick to the term "consistency".

- The random $N \times N$ matrices are drawn from the space of
  - OE: real symmetric matrices. Dyson index $\beta \equiv 1$
  - UE: Hermitean matrices. Dyson index $\beta \equiv 2$
  - SE: quaternion real Hermitean matrices. Dyson index $\beta \equiv 4$.
- The probability $P(H)\mathrm{d}H$ is invariant under the transformation
  - OE: $H \mapsto O^T HO$, O any $N \times N$ real orthogonal matrix
  - UE: $H \mapsto U^{-1}HU$, U any $N \times N$ unitary matrix
  - SE: $H \mapsto W^R HW$, W any $N \times N$ symplectic matrix.
- If used in ordinary quantum mechanics, the system described is[2]
  - OE: invariant under time-reversal and rotations, and has an integer total angular momentum
  - UE: not invariant under time-reversal
  - SE: invariant under time-reversal, but not under rotations, and has a half-integer total angular momentum.

For a definition of a quaternion real matrix, the symplectic group, and W^R, we refer to [9]. If you are already familiar with random matrix theory, you may have noticed that the restriction to Gaussian ensembles has not been imposed. This restriction was originally introduced to ensure that the elements of the random matrices were statistically independent. However, it has been realized [10] that this requirement is not essential, and in fact relaxing it leads to a deeper understanding of the consistency of random matrix predictions.

2.1 Consistency of Random Matrix Theory

The definition of the three random matrix ensembles is given through the partition function

$$\mathcal{Z}_\beta \equiv \int \mathrm{d}HP(H) \equiv \int \mathrm{d}H \, |\det(H)|^\alpha \exp(-\frac{N\beta}{2}\mathrm{Tr}V(H^2)) \ , \tag{2.1}$$

where α is a real parameter ($\alpha > 1/2$), and $V(x)$ is an analytic function, i.e. V has a well defined power series expansion. The integration is over the matrix spaces already introduced and $\mathrm{d}H$ is the associated Haar measure. The invariance of the measure and the cyclic property of the determinant and trace makes it possible to change the integration variables from the matrices to their eigenvalues. The result is (see eg. [9])

$$\begin{aligned}
\mathcal{Z}_\beta &\equiv \int (\prod_{i=1}^{N} \mathrm{d}x_i) P_N(x_1, \ldots, x_N) \\
&\equiv \int (\prod_{i=1}^{N} \mathrm{d}x_i \, |x_i|^\alpha \exp\left(-\frac{N\beta}{2} V(x_i^2)\right)) \prod_{m<n} |x_m - x_n|^\beta \quad .
\end{aligned} \tag{2.2}$$

Here we made explicit the size N of the matrices. Note the appearance of the Vandermonde determinant $\prod_{m<n} |x_m - x_n|^\beta$. As the size N is the matrix analogue of the volume in physical systems, our

[2]For instance, if a quantum mechanical system is invariant under time-reversal and rotations, and has an integer total angular momentum, one can show that the Hamiltonian of such a system is real symmetric. Thus, it is described by the OE.

interest is the correlation functions for the eigenvalues in the $N \to \infty$ limit. The k-point correlation function (1.1) is given by [9]

$$d(x_1, \ldots, x_k) \;\equiv\; \int (\prod_{j=k+1}^{N} \mathrm{d}x_j) P_N(x_1, \ldots, x_N) \;, \tag{2.3}$$

since $d(x_1, \ldots, x_k)\mathrm{d}x_1 \cdots \mathrm{d}x_k$ is the probability of finding an eigenvalue in each of the intervals $[x_j, x_j + \mathrm{d}x_j]$, $j = 1, \ldots, k$. The all important observation at this point is that $d(x_1, \ldots, x_k)$ vanishes whenever two eigenvalues take identical values, due to the Vandermonde determinant. This is *level repulsion.*

There is an amount of arbitrariness in the partition function (2.1,2.2) since the function V is not determined by the symmetry of the physical systems. All that is required from V is that it confines the eigenvalues in such a way that the correlation functions are well defined in the $N \to \infty$ limit. This allow us to formulate a consistency requirement [10]: *The predictions from random matrix theory must be independent of* V. This requirement is met by the spectral correlation functions of random matrix theory only in a very specific limit. This limit is reached when measuring the correlations on the scale of the average level spacing $1/d(E)$. (In the random matrix model defined in (2.1) this correspond to the limit $1/N$ as $N \to \infty$.) This is precisely the same lesson which is to be learned from the observations in physical systems (see e.g. [8]). The correlation functions take a universal form when measured on the scale of $1/d(E)$. On the full scale the correlation functions depend on the details of the model; in the context of random matrix theory, on V.

2.2 The Two-point Function

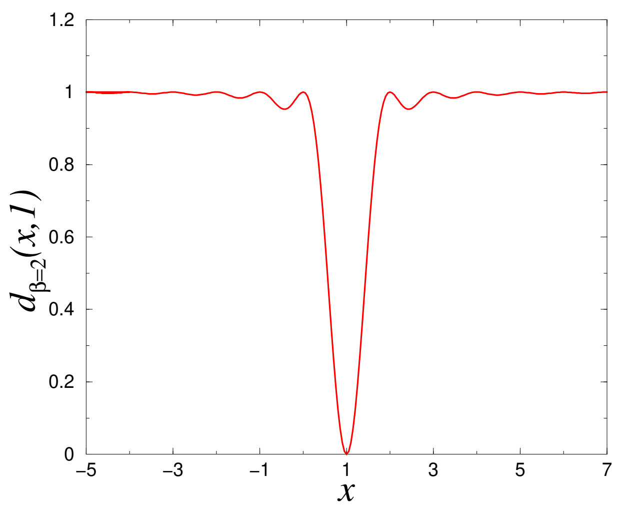

Figure 1: The universal two-point function for the Unitary Ensemble of random matrix theory. Here shown with the second argument fixed to 1. Note that the two point function is translationally invariant and symmetric, since it depends only on the absolute magnitude of the difference between its arguments. Note also that the level repulsion is apparent.

The general expression for the correlation functions of the Unitary Ensemble ($\beta = 2$) of random matrix theory is given compactly in the second reference of [10]

$$d_{\beta=2}(E_1,\ldots,E_k) = \det_{1\le a,b\le k}\left(\frac{\pi}{2}\sqrt{x_a x_b}\frac{J_{\alpha+1/2}(\pi x_a)J_{\alpha-1/2}(\pi x_b) - J_{\alpha-1/2}(\pi x_a)J_{\alpha+1/2}(\pi x_b)}{x_a - x_b}\right) . \tag{2.4}$$

Where the variables have been rescaled $x_i = d(E_i)E_i$ and the limit $N \to \infty$ has been taken in order to reach the universal limit. The symbols J_α denote the Bessel functions of the first kind. Our main interest in this report is the two-point function for $\alpha = 0$. For this value of α and for $\beta = 2$ the two-point function simplifies to

$$d_{\beta=2}(E, E+r) = 1 - \frac{1}{2\pi^2 r^2}(1 - \cos(2\pi r)) = 1 - \left[\frac{\sin(\pi r)}{\pi r}\right]^2 . \tag{2.5}$$

Note that the two-point correlation function is translationally invariant (E independent) and symmetric in r, see also Fig 1. The difference r between the arguments of $d(E, E+r)$ is measured in units of $1/d(E)$. The level repulsion is explicit, $d(E, E+r) \to 0$ as r^2 when $r \to 0$.

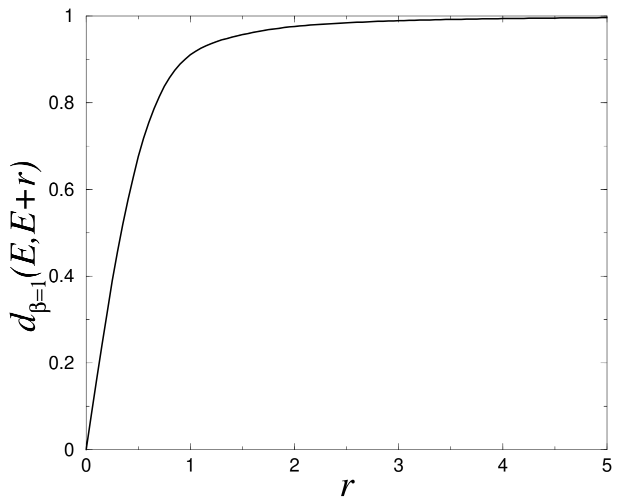

Figure 2: The universal two-point function for the Orthogonal Ensemble of random matrix theory. Shown here is the result of a numerical evaluation of the exact expression (2.6). As for the Unitary ensemble the level repulsion is apparent, but notice that the repulsion in this case is linear, not quadratic.

For later use we also will need the result for the two-point function obtained in the Orthogonal Ensemble ($\beta = 1$) of random matrix theory. The result is not as elegant as (2.5) it reads [9]

$$d_{\beta=1}(E, E+r) = 1 - \left(\int_r^\infty \frac{\sin(\pi t)}{\pi t}\mathrm{d}t\right)\left(\frac{\cos(\pi r)}{r} - \frac{\sin(\pi r)}{\pi r^2}\right) - \left[\frac{\sin(\pi r)}{\pi r}\right]^2 \tag{2.6}$$

Because of the lack of a compact form we will also need the leading small and large r (in the rescaled units) behaviour

$$d_{\beta=1}(E, E+r) = \frac{\pi^2 r}{6} - \frac{\pi^4 r^3}{60} + \frac{\pi^4 r^4}{135} + \cdots \qquad r \ll 1 \tag{2.7}$$

$$d_{\beta=1}(E, E+r) = 1 - \frac{1}{\pi^2 r^2} + \frac{1 + \cos^2(\pi r)}{\pi^4 r^4} + \cdots \quad r \gg 1. \tag{2.8}$$

The fact that the expressions (2.5) and (2.6) are exact, analytical and most importantly consistent predictions of random matrix theory only gives a hint to spectral universality. Even when the spectral correlation functions are found to be shared with some physical systems then random matrix theory does not give a physical explanation of why this is the case. To understand how the observed spectral universality arises we need to study the theories of dynamical systems. Here we shall derive (2.5) and partly (2.6) from periodic orbit theory.

3 The Gutzwiller Trace Formula

The relation between the solutions of the time independent Schrödinger equation and the periodic orbits of the corresponding classical system has been examined by Gutzwiller [6]. Working explicitly in the semiclassical limit he derived a formula expressing the leading order in $\hbar$ of the Green's function as a sum over all periodic orbits

$$\mathrm{Tr}\ G_{\mathrm{sc}}(E) = \mathrm{Tr}\ G_0(E) + \frac{1}{i\hbar}\sum_p \sum_{r=1}^{\infty} \frac{T_p(E)}{|\det(1-\mathbf{J}_p^r(E))|^{1/2}} e^{r\frac{i}{\hbar}S_p(E)} \ . \tag{3.1}$$

We have already encountered one version of this formula, namely in Eq. (1.3). The relation between the two is simple; the density is proportional to the discontinuity of the Green's function across the real-axis[3]

$$\mathrm{ImTr}\ G(E) = -\pi d(E) \ . \tag{3.2}$$

Also the higher order spectral correlation functions can be evaluated using Gutzwiller's trace-formula (3.1). And since our interest is to understand how the universal limit of the two-point function can be calculated using (3.1) it is of importance to recall the precise definitions and assumptions involved in deriving (3.1). On this end we shall also take a close look at the scales involved in the semiclassical limit.

The quantum mechanics in (3.1) enters only through the scale set by $\hbar$ - every ingredient in the trace formula is classical. The periodic orbits are those special classical trajectories which close and retrace themselves. The phase is given by the action S_p for r traversals of the periodic orbit labelled by p

$$rS_p(E) \equiv r\oint_p \mathbf{p}\cdot \mathrm{d}\mathbf{q} \ . \tag{3.3}$$

The period T_p for one traversal is

$$T_p(E) = \frac{\mathrm{d}S_p(E)}{\mathrm{d}E} \ . \tag{3.4}$$

[3]The relation is readily established

$$\begin{aligned}\lim_{\epsilon\to 0}\mathrm{Tr}(G(E-\epsilon)-G(E+\epsilon)) &= \lim_{\epsilon\to 0}\sum_n \frac{1}{E-E_n-i\epsilon} - \frac{1}{E-E_n+i\epsilon} \\ &= \lim_{\epsilon\to 0}\sum_n \frac{-2i\epsilon}{(E-E_n)^2+\epsilon^2} \\ &= -\frac{i}{\pi}\sum_n \delta(E-E_n) \ .\end{aligned}$$

Finally the amplitude $A_p(E) \equiv|\det(1-\mathbf{J}_p^r(E))|^{-1/2}$ is determined by the eigenvalues of the monodromy matrix $\mathbf{J}_p$ for the coordinates transverse to the periodic orbits (see eg. [5]). Note that we have excluded the Maslov index, since this contribution to (3.1) is completely passive in the present context.

A derivation of Gutzwiller's trace formula based on the Wentzel-Kramer-Brillouin ansatz

$$\psi(\mathbf{q},t) = A(\mathbf{q},t)e^{iR(\mathbf{q},t)/\hbar} \tag{3.5}$$

for the solution to the Schrödinger equation given is in [5]. Here we give a brief review, The Wentzel-Kramer-Brillouin decomposition of a complex wave function is completely general. And for a separable Hamiltonian, i.e. when the potential is independent of p, the Schrödinger equation provide two coupled second order differential equations for $A(\mathbf{q},t)$ and $R(\mathbf{q},t)$. In the strict semiclassical limit the dependence on $A(\mathbf{q},t)$ is suppressed in one of the two equations and this equation takes the form of the classical Hamilton-Jacobi equation for the function $R(\mathbf{q},t)$

$$\frac{\partial R}{\partial t} + H(\mathbf{q},\frac{\partial R}{\partial \mathbf{q}}) = 0 \ . \tag{3.6}$$

The Hamilton-Jacobi equation ensures that we can draw on our knowledge from classical analytical mechanics. In this spirit one introduces the action, S, and studies the evolution of the density of state given by the square of A. In order to make progress it it necessary to assume that the functions $A(\mathbf{q},t)$ and $R(\mathbf{q},t)$ are smoothly varying functions. This indirectly justifies the use of a number of stationary phase approximations. The final result of the manipulations is an expression for the propagator as a sum over all classical trajectories from $\mathbf{q}'$ to $\mathbf{q}''$ (see. eg. [5])

$$K_{\rm sc}(\mathbf{q}'',\mathbf{q}',t) \;=\; \sum_j \frac{1}{(2\pi i\hbar)^{d/2}} \,|\det(\frac{\partial \mathbf{p}''}{\partial \mathbf{q}'})_j|^{1/2}\; e^{\frac{i}{\hbar}(R_j(\mathbf{q}'',t)-R_j(\mathbf{q}',0))} \tag{3.7}$$

the Laplace transform of which is the semiclassical Green's function

$$G_{\rm sc}(\mathbf{q}'',\mathbf{q}',E) \;=\; \frac{1}{i\hbar}\int_0^\infty {\rm d}t\; K(\mathbf{q}'',\mathbf{q}',t)\exp(iEt/\hbar) \ . \tag{3.8}$$

Provided that one can find all classical trajectories this expression gives the semiclassical Green's function. Such an explicit evaluation of the summation over every classical orbit from $\mathbf{q}'$ to $\mathbf{q}''$ is however out of the question for the time being. Fortunately, we need only to evaluate the trace of the Greens function in order to calculate the spectral correlation functions, cf. Eq. (3.2).
As is required by the trace in (3.2) we set $\mathbf{q} \equiv \mathbf{q}' = \mathbf{q}''$ and integrate over all possible values of $\mathbf{q}$ in (3.8). The $\mathbf{q}$ integral over (3.8) is evaluated in the stationary phase approximation. The phase of the integrand is the action S, since $S(\mathbf{q}'',\mathbf{q}',t) = R_j(\mathbf{q}'',t) - R_j(\mathbf{q}',0) + Et$. The stationary phase approximation hence impose that the $\mathbf{q}$-variation of the action must vanish

$$\frac{\partial S(\mathbf{q},\mathbf{q},E)}{\partial \mathbf{q}} \equiv \left(\frac{\partial S(\mathbf{q}'',\mathbf{q}',E)}{\partial \mathbf{q}''} + \frac{\partial S(\mathbf{q}'',\mathbf{q}',E)}{\partial \mathbf{q}'}\right)_{\mathbf{q}''=\mathbf{q}'=\mathbf{q}} \equiv \mathbf{p}'' - \mathbf{p}' \ . \tag{3.9}$$

Therefore, of all the classical orbits which take off in $\mathbf{q}' = \mathbf{q}$ and come back $\mathbf{q}'' = \mathbf{q}$, only those which do so smoothly, $\mathbf{p}'' - \mathbf{p}' = 0$, contributes to the sum in (3.8). Thus, the trace of the Green's functions is reduced to a summation over all periodic orbits.

Since we made use of the stationary phase approximation this conclusion is subject to important exceptions. These exceptions arise because $\hbar$ introduces a scale in the exponent of (3.8). For the stationary phase approximation to be valid we must require that $S_j(\mathbf{q}'', \mathbf{q}', E) \gg \hbar$. Hence we must single out short trajectories with $S_j(\mathbf{q}'', \mathbf{q}', E) \leq \hbar$, and treat them independently of the stationary phase approximation. The contribution to Tr $G_{\rm sc}(E)$ from these short trajectories is denoted by Tr $G_0(E)$ in (3.1). This completes our presentation of the arguments leading to Gutzwiller's trace formula.

3.1 The Scales Involved

Even though the real part of Tr $G_0(E)$ is infinite, the imaginary part is finite. Hence it makes sense to use (3.2) to calculate the contribution to the density coming from the short trajectories. The resulting contribution is exactly the same as the one obtained for the average density of states, $\bar{d}(E)$, using the Weyl rule, see eg. [5]. (Weyls rule amounts to counting the available quantum cells in phase space below E.) For a separable Hamiltonian, $\mathbf{H} = \mathbf{p}^2/2m + V(\mathbf{q})$, the result is

$$\begin{aligned} d_0(E) \;=\; \bar{d}(E) \;&=\; \frac{2m}{(2\pi\hbar)^f} \int \mathrm{d}^f\mathbf{p}\ \mathrm{d}^f\mathbf{q}\,\delta(\mathbf{H}(\mathbf{p},\mathbf{q}) - E) && (3.10) \\ &=\; \frac{2m}{\hbar^f 2^f \pi^{f/2}\Gamma(f/2)} \int_{V(\mathbf{q})<E} \mathrm{d}^f\mathbf{q}\ [2m(E - V(\mathbf{q}))]^{f/2}\ , && (3.11) \end{aligned}$$

where f is the number of degrees of freedom. This observation is all important since our interest concerns universal fluctuations in the spectral correlation functions. Recall, from section 2.1, that the universal correlation functions describe the fluctuations on the scale of $1/d(E)$ around the mean spectral correlation function. By the prefactor in Eq.(3.11) we have that $1/d(E)$ is of order $\hbar^f$. The contribution to the trace of the semiclassical Green's function from a orbit of period T is set by $\hbar/T$. This is to be compared with the scale of $1/d(E)$, $T \simeq \hbar^{1-f}$. In conclusion, for systems with more than one degree of freedom ($f > 1$) the contribution to the spectral correlation functions in the semiclassical limit on the scale set by $1/d(E)$ is given by the long orbits $T \simeq \hbar^{1-f}$. In Gutzwiller's trace formula the long orbits are represented in the periodic orbit summation.

4 The Hannay and Ozorio de Almeria Sumrule

The trace formula of Gutzwiller (3.1) reduces the trace of the semiclassical Green's function to a sum over all classical periodic orbits. Finding the infinite family of all periodic orbits of a given system and determining explicitly the periodic orbit sum is however not a trivial task. In fact the number of periodic orbits with period less than some value σ grows exponentially with σ in a classical chaotic system, see for instance [7]. This seemingly devastating fact can actually be turned to our advantage. Hannay and Ozorio de Almeida [11] realized that information about the amplitudes in Gutzwiller's trace formula $A_p(E) \equiv T_p(E)\ |\det(1 - \mathbf{J}^r_p(E))|^{-1/2}$ can be extracted without explicit knowledge about the periodic orbits. This is expressed mathematically through a sumrule for the amplitudes. The sumrule is established under an assumption of ergodicity. Since the general task here is to evaluate the spectral correlation functions of semiclassically chaotic systems through the trace formula the sumrule will prove very useful. In this section we give the derivation of the sumrule originally presented in [11].

Consider a bounded system defined by a Hamiltonian function $\mathbf{H}(\mathbf{r})$, where $\mathbf{r} \equiv (\mathbf{p}, \mathbf{q})$ are the coordinates in the $2f$-dimensional phase space. We want to discuss the evolution of points in phase space and for that we introduce the time average

$$\langle \delta(\mathbf{r} - \mathbf{r}_t) \rangle_t \equiv \lim_{T \to \infty} \frac{1}{2T} \,\bar{\int}_{-T}^{T} \delta(\mathbf{r} - \mathbf{r}_t)\ \mathrm{d}t\ . \tag{4.1}$$

The bar indicates that an arbitrarily small interval around $t = 0$ for technical reasons is excluded from the integral. The average introduced allows for a simple definition of ergodic motion. The bounded system exhibits ergodic motion iff for almost all $\mathbf{r}_0$

$$\langle \delta(\mathbf{r} - \mathbf{r}_t) \rangle_t = \frac{\delta(\mathbf{H}(\mathbf{r}) - \mathbf{H}(\mathbf{r}_0))}{\int \delta(\mathbf{H}(\mathbf{r}) - \mathbf{H}(\mathbf{r}_0))\ \mathrm{d}^{2f}\mathbf{r}_0}\ . \tag{4.2}$$

The formula expresses the ergodic hypothesis: *Almost all initial points* $\mathbf{r}_0$ *have trajectories which fills the entire energy surface.* The points excluded are exactly those lying on the periodic orbits. This is unfortunate in the present context and we therefore reformulate the ergodic hypothesis (4.2). The trick needed is to introduce a smearing of the initial position. This is effectuated by assigning a width ϵ to the delta function peak. The average $\langle \delta(\mathbf{r} - \mathbf{r}_t) \rangle_t$ introduced is modified to

$$\langle \delta(\mathbf{r} - \mathbf{r}_t) \rangle_t \ \equiv\ \lim_{\epsilon \to 0} \lim_{T \to \infty} \frac{1}{2T} \,\bar{\int}_{-T}^{T} \delta_\epsilon(\mathbf{r} - \mathbf{r}_t)\ \mathrm{d}t\ . \tag{4.3}$$

where the subscript ϵ on δ_ϵ indicates the width of the peak. The modified ergodic hypothesis is obtained using this definition of the time average $\langle \delta(\mathbf{r} - \mathbf{r}_t) \rangle_t$ in (4.2). The smearing allows us to include initial points $\mathbf{r}_0$ on periodic orbits; the integral in (4.3) represents the time evolution of a tube initially of radius ϵ. For the consistency of the two definitions of ergodic systems, that is using either (4.1) or (4.3) in (4.2), we must assume that the limits $T \to \infty$ and $\epsilon \to 0$ almost always commute

$$\lim_{\epsilon \to 0} \lim_{T \to \infty} \frac{1}{2T} \,\bar{\int}_{-T}^{T} \delta_\epsilon(\mathbf{r} - \mathbf{r}_t)\ \mathrm{d}t \ =\ \lim_{T \to \infty} \lim_{\epsilon \to 0} \frac{1}{2T} \,\bar{\int}_{-T}^{T} \delta_\epsilon(\mathbf{r} - \mathbf{r}_t)\ \mathrm{d}t\ . \tag{4.4}$$

The message implicit in this assumption is that the long periodic orbits are (almost always) dense in phase space. Dividing (4.4) by the left hand side we have

$$\left\langle \frac{\delta(\mathbf{r} - \mathbf{r}_t)}{\langle \delta(\mathbf{r} - \mathbf{r}_t) \rangle_t} \right\rangle_t \ =\ 1\ . \tag{4.5}$$

The outcome of this manipulation is that the density of the periodic orbits in phase space is uniform when weighted by $\langle \delta(\mathbf{r} - \mathbf{r}_t) \rangle_t$. Note that uniformity holds by assumption. This assumption is referred to as *the principle of uniformity.*

The weighted delta-function, $\delta(\mathbf{r} - \mathbf{r}_t)/\langle \delta(\mathbf{r} - \mathbf{r}_t) \rangle_t$, is a function which picks up only the periodic orbits[4]. As we wish to examine periodic orbits at a fixed energy, E, we are lead to the integral which will give us the Hannay-Ozorio de Almeida sumrule

$$\begin{aligned}
\int \frac{\delta(\mathbf{r} - \mathbf{r}_t)}{\langle \delta(\mathbf{r} - \mathbf{r}_t) \rangle_t} \delta(\mathbf{H}(\mathbf{r}_0) - E)\ \mathrm{d}^{2f}\mathbf{r}_0 \ &=\ \sum_p \left|\frac{\mathrm{d}T_p}{\mathrm{d}E}\right| \int_{\Gamma_p} \frac{\delta(\mathbf{r} - \mathbf{r}_t)}{\langle \delta(\mathbf{r} - \mathbf{r}_t) \rangle_t}\ \mathrm{d}^{2f}\mathbf{r}_0\ \delta(T - T_p) \\
&\equiv\ \sum_p I_p \delta(T - T_p)\ .
\end{aligned} \tag{4.6}$$

Where Γ_p is a domain in phase space enclosing only the p^{th} periodic orbit. To obtain the equality we have made use of the fact that for fixed energy the periods of the periodic orbits form a discrete

[4]The technical problem related to the exclusion of the arbitrarily small region around 0 in (4.1) is related to the cancellations of singularities in this ratio, see [11] for details.

set. The remaining steps of the derivation of the sumrule are first, an evaluation of the time average of (4.6) and second, to express the intensities, I_p, in terms of the amplitude introduced in the trace formula. The time average of (4.6) is readily obtained; using (4.5) the time dependence drops out and the time integral trivially gives $2T$

$$\lim_{T\to\infty} 2T \int \delta(\mathbf{H}(\mathbf{r}_0) - E)\, \mathrm{d}^{2f}\mathbf{r}_0 = \lim_{T\to\infty} \sum_{|T_p|<T} I_p \ . \tag{4.7}$$

To evaluate I_p we make use of the ergodic hypothesis (4.2) to rewrite the definition of I_p

$$\begin{aligned} I_p &\equiv |\frac{\mathrm{d}T_p}{\mathrm{d}E}| \int_{\Gamma_p} \frac{\delta(\mathbf{r}-\mathbf{r}_t)}{\langle\delta(\mathbf{r}-\mathbf{r}_t)\rangle_t}\, \mathrm{d}^{2f}\mathbf{r}_0 \\ &= \int \delta(\mathbf{H}(\mathbf{r}_0) - E)\, \mathrm{d}^{2f}\mathbf{r}_0\, |\frac{\mathrm{d}T_p}{\mathrm{d}E}| \int_{\Gamma_p} \frac{\delta(\mathbf{H}-\mathbf{H}_t)}{\delta(\mathbf{H}-\mathbf{H}_t)}\, \delta(t-T_p)\, \delta(\mathbf{r}-\mathbf{r}_t)\, \mathrm{d}\mathbf{H}\, \mathrm{d}t\, \mathrm{d}^{2(f-1)}\mathbf{r} \ . \end{aligned} \tag{4.8}$$

Note that only the transverse coordinates are included in the lasts phase space integration. The motion along the periodic orbit gives rise to the $\mathbf{H}$ and t integration. Changing variables from T to E in $\delta(t - T_p)$ yields a prefactor which cancels $|\mathrm{d}T_p/\mathrm{d}E|$, leaving the t integration trivial. The resulting $\delta(\mathbf{H} - E)$ fixes the energy through the $\mathrm{d}\mathbf{H}$ integration. We have

$$I_p \equiv \int \delta(\mathbf{H}(\mathbf{r}_0) - E))\, \mathrm{d}^{2f}\mathbf{r}_0\ T_p \int_{\Gamma_p} \delta(\mathbf{r}-\mathbf{r}_T)\, \mathrm{d}^{2(f-1)}\mathbf{r} \tag{4.9}$$

$$= \int \delta(\mathbf{H}(\mathbf{r}_0) - E)\, \mathrm{d}^{2f}\mathbf{r}_0\ T_p\, |\det(1-\mathbf{J}_p)|^{-1} \tag{4.10}$$

In the latter line we gave the second integral in terms of the $2(f-1)\times 2(f-1)$ monodromy matrix $J \equiv \partial\mathbf{r}_T/\partial\mathbf{r}$. This brings us to our final step. Inserting the above expression for I_p into (4.7) we find for large T

$$\sum_{|T_p|<T} T_p\, |\det(1-\mathbf{J}_p)|^{-1} \ \sim\ 2T \ . \tag{4.11}$$

This is the Hannay-Ozorio de Almeida sumrule. It states that the weighted density of periodic orbits $f(t)$ has unity as its limiting value

$$f(t) \equiv \sum_p T_p\, |\det(1-\mathbf{J}_p)|^{-1}\, \delta(t-T_p) \to 1 \ \text{ for } \ T\to\infty \ . \tag{4.12}$$

This can be seen to be equivalent with (4.11) by integration over t.

From the discussion of universality in section 2.1 and energy scales in section 3.1 we expect that the long periodic orbits will build up the correlation functions on the scale where universality is expected. The sumrule just derived apply exactly to the long orbits and will be the main tool when we now rederive the random matrix results from Gutzwiller's trace formula.

5 The Two Point Function in the Diagonal Approximation

The virtue of the Hannay-Ozorio de Almeida sumrule is that we can gain information about the amplitudes without explicit knowledge of the periodic orbits. Imposing the assumption of ergodicity, or hard chaos, ensures the validity of the Hannay-Ozorio de Almeida sumrule. In the following we shall see how this allows us to evaluate the spectral correlation functions in the limit where the functions are expected to take the form given by random matrix theory. In this section we treat the contribution to the two-point function from the diagonal terms. The evaluation of the off-diagonal terms is the topic of the next section.

The definition of the spectral two-point function

$$d(E, E+x) \equiv \frac{1}{d^2(E)} \langle d(E) d(E+x) \rangle_E \tag{5.1}$$

includes an average. In the theory of periodic orbits we consider a single system, and the average is over an energy interval which is long on the scale of $1/d(E)$ but short in the usual units. This average is indicated by the subscript E. To reach the limit where universal correlations are expected we consider values of x of the order $\mathcal{O}(1/d(E))$. In this limit we can replace the factor $1/d^2(E)$ in (5.1) by $1/d_0^2(E)$, cf. section 3.1.

Using the notation, $d(E) = d_0(E) + d_{\rm osc}(E)$, introduced in (1.2) in the definition (5.1) we obtain four terms

$$\langle d(E, E+x) \rangle_E \;=\; \frac{d_0^2(E)}{d_0^2(E)} + \frac{d_0(E)}{d_0^2(E)} \langle d_{\rm osc}(E) + d_{\rm osc}(E+x) \rangle_E + \frac{1}{d_0^2(E)} \langle d_{osc}(E) d_{\rm osc}(E+x) \rangle_E \ . \tag{5.2}$$

The two terms in the middle vanishes since $d_{\rm osc}(E)$ averages to zero. This leaves, using (1.3) to express $d_{\rm osc}(E)$,

$$\langle d(E, E+x) \rangle_E \;=\; 1 + \frac{1}{(d_0 \pi \hbar)^2} \left\langle \sum_{p,q} \sum_{r,u=1}^{\infty} \frac{T_p(E) \cos(r \frac{1}{\hbar} S_p(E))}{|\det(1 - \mathbf{J}_p^r(E))|^{1/2}} \frac{T_q(E) \cos(u \frac{1}{\hbar} S_q(E+x))}{|\det(1 - \mathbf{J}_q^u(E))|^{1/2}} \right\rangle_E \ . \tag{5.3}$$

Where we have neglected the $\mathcal{O}(1/d_0(E))$ difference in the arguments of the two amplitudes and the two periods. Using the cosine relation $\cos(rS_p)\cos(uS_q) = (\cos(rS_p - uS_q) + \cos(rS_p + uS_q))/2$, we obtain a term which does not oscillate as a function of energy when $rS_p - uS_q = 0$ (the former) and an oscillating term (the latter) which averages to zero. Furthermore, since $S_q(E+x) \simeq S_q(E) + x\mathrm{d}S_q/\mathrm{d}x = S_q(E) + xT_q$, see (3.4), we have

$$\langle d(E, E+x) \rangle_E = \tag{5.4}$$
$$1 + \frac{1}{2(d_0 \pi \hbar)^2} \left\langle \sum_{p,q} \sum_{r,u=1}^{\infty} \frac{T_p(E) T_q(E)}{|\det(1 - \mathbf{J}_p^r(E))|^{1/2} |\det(1 - \mathbf{J}_q^u(E))|^{1/2}} \cos\left(\frac{rS_p(E) - uS_q(E)}{\hbar} + \frac{uT_q x}{\hbar}\right) \right\rangle_E \ .$$

As noted above the diagonal terms, $rS_p = uS_q$, in the double sum oscillate less than the rest and seem to have a better chance of surviving the energy average. This motivates a decomposition of the two-point function into a part from the diagonal terms and a part from the off-diagonal terms

$$d(E, E+x) \;\equiv\; 1 + d^{(\rm d)}(E, E+x) + d^{(\rm off)}(E, E+x) \ . \tag{5.5}$$

Here we evaluate the diagonal contribution

$$d^{(\rm d)}(E, E+x) \;=\; \frac{1}{2(d_0 \pi \hbar)^2} \left\langle \sum_p \sum_{r=1}^{\infty} \frac{g_p T_p^2(E)}{|\det(1 - \mathbf{J}_p^r(E))|} \cos\left(\frac{rT_p x}{\hbar}\right) \right\rangle_E \ , \tag{5.6}$$

where the action multiplicities g_p by definition gives the number of times where $rS_p = uS_q$. By choice we work with x fixed in the semiclassical limit, $x = |E_2 - E_1| \; d((E_2 + E_1)/2)$. Therefore, as $\hbar \to 0$, the cosine singles out the orbits with increasingly long period. This hints to the fact that we can make use of the Hannay-Ozorio de Almeida sumrule. To use the sumrule the terms to be summed over must depend only on T_p. In (5.6) the terms also depend on the g_p. Fortunately, for 'typical' systems [3] the long periodic orbits have almost constant g_p. (We return to discuss the class of 'typical' systems in section 7.) We denote the average value of g_p by $\bar{g}$. We assume that we can substitute $\bar{g}$ for g_p and evaluate the sum in (5.6) using the Hannay-Ozorio de Almeida sumrule. First, since the number of the periodic orbits with a period $T_p \in [T, T+\Delta T]$ increases rapidly as T increases, we can neglect the orbits which are long due to repetitions, i.e. we keep only the $r=1$ in the repetition summation. Second, we multiply the sum by a delta function $\delta(t-T_p)$ and integrate over t,

$$d^{(\mathrm{d})}(E,E+x) \sim \frac{\bar{g}}{2(d_0\pi\hbar)^2}\left\langle\int \mathrm{d}t \sum_p \frac{T_p(E)}{|\det(1-\mathbf{J}_p(E))|}\delta(t-T_p)t\cos(\frac{tx}{\hbar})\right\rangle_E . \tag{5.7}$$

The first part of the sum is now recognised as the weighted density $f(t)$ introduced in (4.12). The Hannay-Ozorio de Almeida sumrule state that $f(t)=1$ so

$$d^{(\mathrm{d})}(E,E+x) \sim \frac{\bar{g}}{2(d_0\pi\hbar)^2}\left\langle\int \mathrm{d}t\; t\cos(\frac{tx}{\hbar})\right\rangle_E . \tag{5.8}$$

This is a standard integral and we find

$$d^{(\mathrm{d})}(E,E+x) \sim -\frac{\bar{g}}{2\pi^2x^2d_0^2} . \tag{5.9}$$

The average value $\bar{g}$ of g_p which approximates the true value for the long orbits is determined by the symmetries of the system. In the case where there is no symmetries $\bar{g}=1$, while in the case where there is a two-fold symmetry, like time reversal symmetry, $\bar{g}=2$. Using these values of $\bar{g}$ we can compare the diagonal approximation, $d(E,E+x) \sim 1+d^{(\mathrm{d})}(E,E+x)$, with the random matrix prediction. For $\bar{g}=2$ we expect the statistics of the Orthogonal Ensemble and for $\bar{g}=1$ we expect that the Unitary Ensemble applies. Comparing (5.7) and the large x expansions (2.8) and (2.5) we find that the diagonal approximation is consistent with the leading order non-oscillatory part of the random matrix prediction.

6 Beyond the Diagonal Approximation

The results from the diagonal approximation are promising: The correlations functions derived are consistent with the random matrix correlation functions. The diagonal approximation does however not reproduce the full random matrix expressions. Going beyond the diagonal approximation Bogomolny and Keating [3] showed that also the leading order oscillatory part of the random matrix prediction is reproduced by the trace formula. The arguments leading to this conclusion are somewhat twisted. It is, at present, not possible to evaluate $d^{(\mathrm{off})}(E,E+x)$ directly. Here we present an indirect method suggested in [3].

The problem in evaluating the off-diagonal terms in (5.4) arises because these terms depend on the correlation between the action and amplitudes of different orbits, cf. (5.4). For a general chaotic system we have no knowledge about these effects. The trick of [3] is to consider a truncated version of Gutzwiller's trace formula where only orbits with periods T_p less than a certain value, T^*, are included. By definition,

$$d_{T^*}(E) \equiv \bar{d}(E) + \frac{1}{\pi\hbar}\mathrm{Re}\sum_{rT_p<T^*}\frac{T_p(E)}{|\det(1-\mathbf{J}_p^r(E))|^{1/2}}\exp\left(\frac{i}{\hbar}rS_p(E)\right) . \tag{6.1}$$

Just as Gutzwiller's trace formula gives approximations to the spectrum of quantum systems, so does the truncated version. In order to obtain these approximate eigenvalues we need the function which counts the number of approximate energy levels below the energy E

$$N_{T^*} \equiv \int_0^E d_{T^*}(E')\,\mathrm{d}E' \ . \tag{6.2}$$

The approximate energy levels, $E_n(T^*)$, are now given by

$$N_{T^*}(E_n(T^*)) \ \equiv \ n + \frac{1}{2} \tag{6.3}$$

(This quantisation condition was introduced and discussed in [12].) The function $d_{T^*}(E)$ measures the exact density of states smoothed on the scale of $\mathcal{O}(\hbar/T^*)$, as can be seen by a simple dimensional argument. We shall take T^* to infinity in the semiclassical limit, e.g. $T^*(E) \simeq 2\pi\hbar\bar{d}(E)$. In order to measure the fluctuations on the scale of $1/d_{T^*}(E)$ scale we need the density, $D_{T^*}(E)$, of the approximate energy levels, $E_n(T^*)$

$$D_{T^*}(E) \ \equiv \ \sum_n \delta(E - E_n(T^*)) \ . \tag{6.4}$$

It is this density which we shall use to define the two-point function

$$d(E+x_1, E+x_2) \ \equiv \ \langle D_{T^*}(E+x_1) D_{T^*}(E+x_2)\rangle_E \ . \tag{6.5}$$

We shall use the counting function N_{T^*} to evaluate the two-point function. First, the two parts of the density (6.1) define a natural separation of the counting function through (6.2)

$$N_{T^*}(E) \ \equiv \ N_0(E) + N_{T^*}^{(\mathrm{osc})}(E) \ . \tag{6.6}$$

Second, by the truncated trace formula (6.1) we have (recall that $T_p = \mathrm{d}S_p/\mathrm{d}E$)

$$N_{T^*}^{(\mathrm{osc})}(E) \ = \ \frac{1}{\pi}\sum_{rT_p<T^*} \frac{1}{r}\frac{1}{|\det(1-\mathbf{J}_p^r(E))|^{1/2}} \sin\left(\frac{1}{\hbar} r S_p(E)\right) \ . \tag{6.7}$$

Now, $N_{T^*}(E)$ is monotonically increasing and by (6.2) the derivative of $N_{T^*}(E)$ is $d_{T^*}(E)$. Hence we can use the standard method of changing variables in a delta function, combined with (6.3), to obtain

$$\delta(N_{T^*}(E) - n - \frac{1}{2}) \ = \ \frac{1}{N'_{T^*}(E)}\delta(E - E_n(T^*)) \ = \ \frac{1}{d_{T^*}(E)}\delta(E - E_n(T^*)) \ . \tag{6.8}$$

Inserting this into (6.4) we find

$$\begin{aligned} D_{T^*}(E) \ &= \ d_{T^*}(E)\sum_n \delta(N_{T^*}(E) - n - \frac{1}{2}) && (6.9)\\ &= \ d_{T^*}(E)\sum_{k=-\infty}^{\infty} (-1)^k \exp(2\pi i k N_{T^*}(E)) \ . && (6.10) \end{aligned}$$

The latter equality is justified by the principle of destructive interference; only if $N_{T^*}(E)$ takes values $n+\frac{1}{2}$ then the phase is stationary and the sum non-zero. Substituting this form of $D_{T^*}(E)$ in (6.5) we get

$$\begin{aligned} d(E+x_1, E+x_2) \ &= \ \langle d_{T^*}(E+x_1) d_{T^*}(E+x_2) && (6.11)\\ &\quad * \sum_{k_1,k_2=-\infty}^{\infty} (-1)^{k_1-k_2} \exp[2\pi i(k_1 N_{T^*}(E+x_1) - k_2 N_{T^*}(E+x_2))]\rangle_E \ . \end{aligned}$$

This is the expression on which all the subsequent is based. We shall now show that the term in (6.11) for which $k_1 = k_2 = 0$ corresponds to the diagonal terms evaluated in the previous section. The remaining terms are therefore identified with the off-diagonal contribution.

6.1 The Term $k_1 = k_2 = 0 \sim$ Diagonal Contribution

We choose to write $d_{T^*}(E) = d_0(E) + d_{T^*}^{(\mathrm{osc})}(E)$ where the terms are identified through the trace formula (6.1) defining $d_{T^*}(E)$. With this notation the term in (6.11) with $k_1 = k_2 = 0$ is

$$\begin{aligned} \langle d_{T^*}(E+x_1) d_{T^*}(E+x_2)\rangle_E &= d_0^2(E) + \langle d_0(E) d_{T^*}(E+x_2) + d_{T^*}(E+x_1) d_0(E)\rangle_E \\ &\quad + \langle d_{T^*}(E+x_1) d_{T^*}(E+x_2)\rangle_E \ . \end{aligned} \tag{6.12}$$

The two terms in the middle have zero averages. Using the definitions (6.2) and (6.6) this leaves

$$\langle d_{T^*}(E+x_1) d_{T^*}(E+x_2)\rangle_E = d_0^2(E) + \left\langle d_{T^*}^{(\mathrm{osc})}(E+x) d_{T^*}^{(\mathrm{osc})}(E+y)\right\rangle_E \ . \tag{6.13}$$

The product $d_{T^*}^{(\mathrm{osc})}(E+x) d_{T^*}^{(\mathrm{osc})}(E+y)$ involves a double sum over periodic orbits with period less than T^*. In order to evaluate this sum we will make use of the diagonal approximation. In the limit $T^* \to \infty$ the double summations are effectively infinite sums. Therefore we are faced with exactly the same double sum as in the previous section. And we can simply reuse the arguments

$$\begin{aligned} \langle d_{T^*}(E+x_1) d_{T^*}(E+x_2)\rangle_E &\simeq d_0^2(E) + \frac{1}{2(d_0\pi\hbar)^2}\left\langle \sum_p \sum_{r=1}^{\infty} \frac{g_p T_p^2}{|\det(1-\mathbf{J}_p^r(E))|} \cos(\frac{rT_p x}{\hbar})\right\rangle_E \\ &\simeq d_0^2(E) + \frac{\bar{g}}{2(d_0\pi\hbar)^2}\left\langle \sum_p T_p \frac{T_p(E)}{|\det(1-\mathbf{J}_p(E))|} \cos(\frac{T_p x}{\hbar})\right\rangle_E \\ &\simeq d_0^2(E) + \frac{\bar{g}}{2(d_0\pi\hbar)^2}\left\langle \int \mathrm{d}t \sum_p \frac{T_p(E)}{|\det(1-\mathbf{J}_p(E))|} \delta(t-T_p) t \cos(\frac{tx}{\hbar})\right\rangle_E \\ &\simeq d_0^2(E) + \frac{\bar{g}}{2(d_0\pi\hbar)^2}\left\langle \int \mathrm{d}t\ t \cos(\frac{tx}{\hbar})\right\rangle_E \\ &\simeq d_0^2(E) - \frac{\bar{g}}{2d_0\pi^2 x^2} \ . \end{aligned} \tag{6.14}$$

Just as in the previous section we have assumed that the g_p of the long orbits can be replaced by a constant $\bar{g}$.

As promised we have shown that the $k_1 = k_2 = 0$ term in (6.11) coincides with the large x leading order non-oscillatory part of the exact random matrix expressions.

6.2 The Remaining Terms $\sim$ Off-Diagonal Contribution

We will now show that the remaining terms in (6.11) give the large x leading order oscillatory part which was not reproduced by the diagonal approximation. To evaluate the terms where k_1 and k_2 are non-zero we draw attention to the exponent in (6.11). Inserting the decomposition (6.6) we find that each term in the sum is a product of two exponentials

$$\exp[2\pi i(k_1 N_0(E+x_1) - k_2 N_0(E+x_2))] \exp[2\pi i(k_1 N_{T^*}^{(\mathrm{osc})}(E+x_1) - k_2 N_{T^*}^{(\mathrm{osc})}(E+x_2))] \ .$$

When performing the energy average the first factor oscillates unless $k_1 = k_2$. Therefore only terms with $k_1 = k_2$ survive the average. Using the definition (6.2)

$$N_0(E + x_1) - N_0(E + x_2) \simeq \frac{\mathrm{d}N_0(E)}{\mathrm{d}E}(x_1 - x_2) = d_0(E)(x_1 - x_2) \tag{6.15}$$

we find for the part of the sum in (6.11) with $k_1 = k_2 \neq 0$

$$d_0^2(E) \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{(2\pi k)^2} \exp[2\pi i d_0 k(x_1 - x_2)] \langle \exp[2\pi i k(N_{T^*}^{(\mathrm{osc})}(E + x_1) - N_{T^*}^{(\mathrm{osc})}(E + x_2))] \rangle_E \ . \tag{6.16}$$

The average of the exponential again involves correlations between the action and the amplitudes of the periodic orbits. In fact, using the power series expansion for the exponential function

$$\exp[x] = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{6.17}$$

is it obvious that these correlations are more extensive than the ones we encountered in (5.4). To proceed we need to assume that the periodic orbits with periods up to T^* can be treated as statistically independent. This key assumption is essentially an extended version of the diagonal approximation. However, introducing it at this late state allow us to evaluate the remaining terms in (6.11) and hence to go beyond the diagonal approximation.
By the assumption the terms in the exponent of (6.16) are statistically independent. Hence we have to good approximation[5]

$$\begin{aligned}
&\langle \exp[2\pi i k(N_{T^*}^{(\mathrm{osc})}(E + x_1) - N_{T^*}^{(\mathrm{osc})}(E + x_2))] \rangle_E \\
&\qquad \simeq \exp[-2\pi^2 k^2 \langle \left(N_{T^*}^{(\mathrm{osc})}(E + x_1) - N_{T^*}^{(\mathrm{osc})}(E + x_2)\right)^2 \rangle_E] \qquad (6.20) \\
&\qquad \simeq \left(\frac{\exp[4\pi^2 \langle N_{T^*}^{(\mathrm{osc})}(E + x_1) N_{T^*}^{(\mathrm{osc})}(E + x_2) \rangle_E]}{\exp[4\pi^2 \langle (N_{T^*}^{(\mathrm{osc})}(E))^2 \rangle_E]} \right)^{k^2} \qquad (6.21)
\end{aligned}$$

The arguments of the exponentials in both the numerator and the denominator once again have correlations between the action and the amplitude. Therefore, in order to evaluate the numerator and the denominator we need to rely on the diagonal approximation and the Hannay-Ozorio de Almeida sumrule. It is convenient first to take a look at the denominator. Again, we need to assume that the repetition sum can be ignored and that the action multiplicities, g_p, can be replaced by their average value $\bar{g}$. With these restrictions we may use the Hannay-Ozorio de Almeida sumrule

$$\exp[4\pi^2 \langle (N_{T^*}^{(\mathrm{osc})}(E))^2 \rangle_E] \sim \exp[\frac{4\pi^2}{2\pi^2} \bar{g} \log T^*] \sim (T^*)^{2\bar{g}} \tag{6.22}$$

[5]For a Gaussian distributed function, $G(E)$, with zero mean the approximation used

$$\langle \exp[iG(E)] \rangle \ \simeq \ \exp[-\frac{1}{2} \langle G^2(E) \rangle] \tag{6.18}$$

is an identity. Proof:

$$\int_{-\infty}^{\infty} \mathrm{d}x \ e^{ix} \frac{1}{\sqrt{\pi}} e^{-x^2} \ = \ \exp[-\int_{-\infty}^{\infty} \mathrm{d}x \ x^2 \frac{1}{\sqrt{\pi}} e^{-x^2}/2] \ . \tag{6.19}$$

From this we have that the terms in the sum in (6.16) decrease as $(T^*)^{-2gk^2}$ in the large T^* limit, and hence that the $k_1 = k_2 = \pm 1$ terms dominate the sum in (6.16). Keeping only these terms, we have for the contribution to (6.11) for $k_1 = k_2 \neq 0$

$$\begin{aligned} & d_0^2(\exp[2\pi i d_0(x_1 - x_2)] + \exp[-2\pi i d_0(x_1 - x_2)]) \exp[4\pi^2 \langle N_{T^*}^{(\mathrm{osc})}(E + x_1) N_{T^*}^{(\mathrm{osc})}(E + x_2)\rangle_E] \\ = \; & d_0^2 \cos(2\pi d_0(x_1 - x_2)) \exp[4\pi^2 \langle N_{T^*}^{(\mathrm{osc})}(E + x_1) N_{T^*}^{(\mathrm{osc})}(E + x_2)\rangle_E] \; . \end{aligned} \tag{6.23}$$

To evaluate the remaining exponential (from in the numerator of (6.21)) we once again turn to the Hannay-Ozorio de Almeida sumrule.

$$\begin{aligned} \exp[4\pi^2 \langle N_{T^*}^{(\mathrm{osc})}(E + x_1) N_{T^*}^{(\mathrm{osc})}(E + x_2)\rangle_E] &= \exp[4\pi^2 \int \mathrm{d}x_1\, \mathrm{d}x_2 \langle d_{T^*}^{(\mathrm{osc})}(E + x_1) d_{T^*}^{(\mathrm{osc})}(E + x_2)\rangle_E] \\ &\sim \exp[-4\pi^2 \int \mathrm{d}x_1\, \mathrm{d}x_2 \frac{\bar{g}}{2\pi^2 (x_1 - x_2)^2}] \\ &\sim \exp[-2\bar{g} \log(x_1 - x_2)] \\ &\sim (x_1 - x_2)^{-2\bar{g}} \end{aligned} \tag{6.24}$$

Combining this with (6.23) we finally have the evaluation of the off diagonal terms (recall, $T^* \sim \hbar d_0$)

$$C_{\bar{g}} \frac{\cos(2\pi d_0(x_1 - x_2))}{(x_1 - x_2)^{2\bar{g}} d_0^{2\bar{g}-2}} \; . \tag{6.25}$$

Up to the constant $C_{\bar{g}}$ this coincides exactly with the leading order oscillatory term of the random matrix. For the Unitary Ensemble the first order expansion of the two point function is exact, cf. (2.5). This allows us to fix the constant $G_{\bar{g}=1}$ by the requirement that $d(E, E + x) \to 0$ as $x \to 0$. When the system enjoys time reversal symmetry we have $\bar{g} = 2$ and the corresponding random matrix ensemble is the Orthogonal one. Here we have no chance of fixing the constant. The present result only allows us to extract the oscillating cosine function and the x^{-4} behaviour. Even with this drawback Bogomolny and Keating conclude [3]: The discrepancy between the two-point function of random matrix theory and that obtained through the diagonal approximation is an artifact of the diagonal approximation and not a physical effect. The leading order oscillatory terms are given by the off-diagonal terms.

7 Discussion

Gutzwiller's trace formula allows us to relate the density of states to the periodic orbits of a system. For a general chaotic system we have no a priori knowledge of either. So in order to prove spectral universality of general chaotic systems using the trace formula we need to supply additional information about the periodic orbits. The additional input comes from assuming the principle of uniformity. This principle lead us to the Hannay-Ozorio de Almeida sumrule. The sumrule fits in naturally in the evaluation of the spectral two-point function in the diagonal approximation. In this approximation the leading order non-oscillatory terms of the random matrix predictions are recovered. Bogomolny and Keating [3] have suggested a way in which also the off-diagonal contribution may be evaluated. The general arguments of [3] reviewed in section 6.2 are admittedly not as direct as one might wish. The real weakness is that it is not possible to evaluate the error made in the various approximations. However, there are examples which allow for a direct approach. For the Riemann ζ-function the off-diagonal terms can be evaluated explicitly [3]. The result agrees with the one obtained through the indirect approach of [3], and hence it provides an important consistency check.

In principle, one could derive the corrections to higher powers in $\hbar$ and in this way go beyond the semiclassical regime where Gutzwiller's trace formula holds. This fact poses the question: Is the semiclassical limit needed to recover the universal spectral statistics prototyped by random matrix theory; will further quantum effects be inconsistent with universality? The answer is probably no. The universality is a result of ergodicity and the quantum effects will most likely enhance the ergodicity. It is of course important that we have used the $\hbar \to 0$ to separate the scales of the universal fluctuations around the overall variations of the spectral correlation functions. As an encouragement to go beyond the leading order we mention that the higher order terms in (2.8) are expected to be protected by universality. Furthermore examining, if the series ends for non-symmetric systems and if it is infinite for time reversal invariant systems, as in random matrix theory, will provide novel insight.

In conclusion we have given evidence that ergodic, i.e. hard chaotic, quantum systems in their semiclassical limit have spectral correlation functions which on the scale $1/d(E)$ coincide with those of random matrix theory. There are, however, a few non-generic examples of ergodic systems which can be shown *not* to share spectral correlation functions with random matrix theory[6]. This observation is crucial, since it sets a challenge to be met by a theory which seeks to explain how spectral universality arises in chaotic dynamical systems. Universality can not be a consequence only of ergodicity. The explanation examined in the present report meets this challenge. In order to evaluate Gutzwiller's trace formula using the Hannay-Ozorio de Ameida sumrule we had to assume that the action multiplicities, g_p, are approximately constant for long periodic orbits. We have used the word 'typical' for systems for which this approximation holds. Now, the point is that all the non-generic examples (ergodic systems with non-random matrix statistics) have a large spread of action multiplicities, g_p, for long periodic orbits. Hence the line of argument presented breaks down and we avoid a contradiction.

As mentioned in the introduction the work presented here can be seen as part of the general quest to describe how spectral universality arises. It is worthwhile to give a comparison of the different approaches followed. Besides the use of trace formulas the line of research goes in two directions. These two are disorder models and chiral perturbation theory. Using the first the aim is, exactly as we have done using the periodic orbit theory, to describe spectral universality of chaotic dynamics. Using the latter the aim is to describe how spectral universality arises in the framework of quantum chromodynamics. Though the areas are quite separate the technical methods used are quite similar. Both methods may be treated using a super symmetric approach [14][2]. It should be emphasised that this is just a technical tool and not a physical supersymmetry, one can avoid using the super symmetric approach using for instance the replica method, cf. the last entry of [2]. Most important in the present context, however, is the fact that both the disorder models and chiral perturbation theory introduce averages over different realizations of the system involved. For that reason they both differ fundamentally from the periodic orbit theory. This difference is realized explicitly in the averages introduced in the definition of the spectral correlation functions. The appearance of the non-'typical' cases is for this reason not explicit in the approach of disorder models. To avoid a contradiction in the line of argument one must argue that the non-'typical' cases have measure zero in the weighted average over all realizations of the systems [15].

[6] An example is given by the cat map

$$\begin{pmatrix} q_{n+1} \\ p_{n+1} \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} q_n \\ p_n \end{pmatrix} \qquad (n, m \mod 1)\ , \tag{7.1}$$

with t_{ij} being integers satisfying $|t_{11} + t_{22}| > 2$ and $t_{11}t_{22} - t_{21}t_{12} = 1$. The problem can be treated exact analytically and the result is [13] that though maximally chaotic, the spectral statistics does not fall into any of the three universality classes prototyped by the random matrix ensembles.

Acknowledgement: I am grateful to S.H. Svendsen for correcting misprints and abuse of language in this report.

References

[1] C.E. Porter, Statistical Theories of Spectra: Fluctuations, Perspectives in Physics, *Academic Press, Inc.*, 1965.

[2] J.C. Osborn, D. Toublan and J.J.M. Verbaarschot, Nucl. Phys. **B540** (1999) 317.
P.H. Damgaard, J.C. Osborn, D. Toublan and J.J.M. Verbaarschot, Nucl. Phys. **B547** (1999) 305.
D. Toublan and J.J.M. Verbaarschot, hep-th/9904199.
P.H. Damgaard and K. Splittorff, hep-th/9912146.

[3] E.B. Bogomolny and J.P. Keating, Phys. Rev. Lett. **77** (1996) 1472.
J.P. Keating, NATO ASI Series B: Physics Vol. 370, *eds.* Lerner, Keating, and Khmelnitskii (1999) 1.

[4] A.V. Andreev, B.D. Simons, O. Agam, and B.L. Altshuler, Nucl. Phys. **B482** (1996) 536; Phys. Rew. Lett. **76** (1996) 3947.

[5] P. Cvitanović, et al., `http://www.nbi.dk/ChaosBook/`, "Classical and Quantum Chaos: A Cyclist Treatice", The Niels Bohr Institute, Copenhagen (2000).

[6] M.C. Gutzwiller, J. Math. Phys. **8** (1967) 1979.
M.C. Gutzwiller, J. Math. Phys. **10** (1969) 1004.
M.C. Gutzwiller, J. Math. Phys. **11** (1970) 1791.
M.C. Gutzwiller, J. Math. Phys. **12** (1971) 343.

[7] M.C. Gutzwiller, "Chaos in Classical and Quantum Mechanics", *Springer-Verlag New York Inc.* (1990).

[8] T. Guhr, A. Mueller-Groeling, and H.A. Weidenmüeller, Phys. Rept. **299** (1998) 189-425.

[9] M.L. Mehta, Random Matrices, *Academic Press, Inc.* , Second edition, 1991.

[10] E. Brezin, A. Zee, Nucl. Phys. **B402** (1993) 613.
G. Akemann, P.H. Damgaard, U. Magnea and S. Nishigaki, Nucl. Phys. **B487** (1997) 721.
P.H. Damgaard and S.M. Nishigaki, Nucl. Phys. **B518** (1998) 495.
M.K. Şener and J.J.M. Verbaarschot, Phys. Rev. Lett. **81** (1998) 248.

[11] J.H. Hannay and A.M. Ozorio De Almeida, J. Phys. A:Math. Gen. **17** (1984) 3429.

[12] R. Aurich, C. Matthies, M. Sieber, and F. Steiner, Phys. Rev Lett. **68** (1992) 1629.
E. Bogomolny and C. Schmit, Nonlinearity **6** (1993) 523.

[13] J.P. Keating, Nonlinearity **4** (1991) 309.

[14] K. Efetov, "Supersymmetry in Disorder and Chaos", *Cambridge University Press* (1997).

[15] M.R. Zirnbauer, NATO ASI Series B: Physics Vol. 370, *eds.* Lerner, Keating, and Khmelnitskii (1999) 153.