Modeling Atomic, Dynamical Friction Using an Inelastic Impact Oscillator Linear Stability Matrix

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July 26, 2004

Abstract

I created a new document for this section because it made it a lot easier to edit.

1 Linear Stability

The linear stability matrix is an important tool for understanding the system. It's eigenvalues and eigenvectors allow one to understand the strength and direction of the stable and unstable manifolds. The matrix will be used in periodic orbit theory, and to find the Lyapunov exponents of the system. The linear stability matrix is defined as $J \equiv \frac{\partial x_i}{\partial x_j}$. As stated above, our system is a combination of two distinct behaviors: one where the particle is above the surface, and one where the particle is interacting with the surface. This implies a natural separation of the matrix into one matrix found through the equations above the surface, and one found by analyzing the behavior while interacting with the surface. This will mean matrix equation will look like

$$\delta \vec{q}_{i+1}^{+} = J \vec{q}_i^{+} = J_I \ J_T \delta \vec{q}_i^{+} \tag{1}$$

where

$$\delta \vec{q}_i^{+} = \begin{pmatrix} \delta x_i^{+} \\ \delta u_i^{+} \\ \delta y_i^{+} \\ \delta v_i^{+} \end{pmatrix} \tag{2}$$

If we then need to change cooorditates to simplfy a matrix, we simply surround the J by 4x4 rotations matricies about the angle. Our equations will then be:

$$\vec{q}_{i+1}^{+} = R^{-1} \ J_I \ R \ J_T \ \vec{q}_i^{+} \tag{3}$$

1.1 Stability Matrix in Transit

We begin by studing the linear stability matrix associated with the particle not interacting with the surface. Recalling the equations of the particle's trajectory, given by Equations [?] and [?]. These, of course, can be easily separated to:

$$
\begin{aligned}
x(t) &= \frac{(u_i^+ - v_c)}{\omega_x}\ \sin(\omega_x t) + x_i^+\ \cos(\omega_x t) + v_c t \\
u(t) &= (u_i^+ - v_c)\ \cos(\omega_x t) - \omega_x x_i^+\ \sin(\omega_x t) + v_c \\
y(t) &= \frac{v_i^+}{\omega_y}\ \sin(\omega_y t) + y_i^+\ \cos(\omega_y t) \\
v(t) &= v_i^+\ \cos(\omega_y t) - \omega_y y_i^+ \sin(\omega_y t)
\end{aligned}
$$

Because they are uncoupled, the natural choice of basis for the linear stability matrix would be the coordinates used to describe these oscilations. x is the horizontal across the averaged surface, y is normal to the averaged surface, and u, v are their respective velocities. Using the definition of the matrix, and the time of flight between two impacts to be τ then the matrix is:

$$
J_T = \begin{pmatrix} \cos(\omega_x\tau) & \frac{\sin(\omega_x\tau)}{\omega_x} & 0 & 0 \\ -\omega_x \sin(\omega_x\tau) & \cos(\omega_x\tau) & 0 & 0 \\ 0 & 0 & \cos(\omega_y\tau) & \frac{\sin(\omega_y\tau)}{\omega_y} \\ 0 & 0 & -\omega_y \sin(\omega_y\tau) & \cos(\omega_y\tau) \end{pmatrix} \tag{4}
$$

1.2 Impact Stability Matrix

For the impact matrix, J_I, we could choose 3 different bases, each with their own advantages. One could take the same basis as the transitive matrix, which would get rid of the need for rotations. Another possibility would be to take the coordinates as along the velocity flow, and normal to the velocity flow. This has rather rancid rotations and the matrix itself is quite non-trivial. The third, and most natural choice is to take the coordinates as normal and tangential to the surface. This choice is recomended by Maťyaś and Klages [[1]] and others. For the fun of it, we will be investigating the latter two matricies.

Since the equation for the velocity just after the impact is given by $\vec{p_i^+} = \vec{p_i^-} - (1+\gamma)p_i^- \cdot \hat{n}$ where $\hat{n}$ is the direction normal to the surface, this will be true no matter the choice of coordinates. If the particle impacts the surface with and angle ϕ, its tangential component will be given by $v^- \sin\phi\hat{t}$ and normal component will be $v^- \cos\phi\hat{n}$. After the impact only the normal component will be affected by the COR, γ. This implies after the impact, the tangential component will be the same, but the normal component of the velocity will be $v^+ \cos\phi'\hat{n}$. The magnitude of the velocity is then given by:

$$v^+ = v^- \sqrt{\sin^2\phi + \gamma^2 \cos^2\phi}$$

.

The angle of the particle leaving the surface is another important generalized quantity. We know that

$$\begin{aligned} -\gamma v_\perp^- + v_{||}^- &= v_\perp^+ + v_{||}^+ \\ \Rightarrow -\gamma v^- \cos\phi &= v^+ \cos\phi\prime \\ v^- \sin\phi &= v^+ \sin\phi\prime \end{aligned} \tag{5}$$

$$\begin{aligned} \Rightarrow \phi\prime &= \cos^{-1}\left(\frac{-\gamma\cos\phi}{\sqrt{\sin^2\phi + \gamma^2\cos^2\phi}}\right) \\ \phi\prime &= \sin^{-1}\left(\frac{\sin\phi}{\sqrt{\sin^2\phi + \gamma^2\cos^2\phi}}\right) \\ \phi\prime &= \tan^{-1}\left(-\frac{1}{\gamma}\tan\phi\right) \end{aligned} \tag{6}$$

Another coordinate invarient property of the system is the normal and tangential components of the surface at the impact. We know the table behaves like $h(x) = A\sin(\omega x + \delta)$, which implies that $\frac{dh(x)}{dx} = \omega A \cos(\omega x + \delta)$. Now because the dynamics refresh every impact, and the impact takes place when the particle is on the surface, then $h(x_i^+) = y_i^+ = A\sin(\omega x_i^+ + \delta)$, which implies the phase at each impact is given by $\delta = \sin^{-1}(\frac{y_i^+}{A}) - \omega x_i^+$. We also know the horizontal position at the next impact from the equation above: $x(\tau)$. Substituting these into h(x) gives the table's position at the next impact,

$$h(\tau) = A\sin(\omega(\frac{(u_i^+ - v_c)}{\omega_x}\ \sin(\omega_x\tau) + x_i^+\ \cos(\omega_x\tau) + v_c\tau) + \sin^{-1}(\frac{y_i^+}{A}) - \omega x_i^+)$$

The direction tangent to the surface, $\hat{t}$, at any point is given by $\tan^{-1}(\frac{dh}{dx})$ which means that at the i^{th} impact, $\hat{t} = \tan^{-1}(\omega A\cos(\omega x_i^- + \delta))$ and $\hat{n} = \hat{t} + \frac{\pi}{2}$.

Now that we have all the preliminary quantities out of the way, we shall look at the two impact stability matricies in the different coordinate frames.

1.3 Local Reference Frame Matrix

One set of generalized coordinates are found by going to the local frame by using the coordinate variation along the flow and normal to the flow: $(\delta q_{||}, \delta v_{||}, \delta q_{\perp}, \delta v_{\perp})$. This implies a rotation about the angle $\theta = \tan^{-1}(\frac{v_y(\tau)}{v_x(\tau)})$. These values are known, but they are computationally non-trivial.

I will now go through each of the derivations of the non-trivial (hense, non-zero) elements of the stability matrix.

J_{11}:$\delta q_{||}^+ = v^+\tau = v^+\frac{\delta q_{||}^-}{v^-} = \sqrt{\sin^2\phi + \gamma^2\cos^2\phi}\delta q_{||}^-$

J_{13} and J_{33}: Upon impact, the particles separated by $\delta q_{\perp}^-$ will reduce their varience normal to the flow, but will skew in such a way that a new varience will be established along the flow. Referencing the picture:

$$
\begin{aligned}
\frac{\delta q_{\perp}^-}{\cos\phi} &= -\frac{\Delta q}{\sin\phi} \\
\Rightarrow \Delta q &= \delta q_{\perp}^- \tan\phi \\
\delta q_{||}^+ &= \Delta q(1 - \frac{v^+}{v^-}) \\
\therefore \delta q_{||}^+ &= \delta q_{\perp}^- \tan\phi(1 - \sqrt{\sin^2\phi + \gamma^2\cos^2\phi})
\end{aligned}
$$

We also find:

$$
\begin{aligned}
\frac{\delta q_{\perp}^+}{\cos\phi\prime} &= -\frac{\delta q_{\perp}^-}{\cos\phi} \\
\Rightarrow \delta q_{\perp}^+ &= -\frac{\gamma}{\sqrt{\sin^2\phi + \gamma^2\cos^2\phi}}\delta q_{\perp}^-
\end{aligned}
$$

J_{22}, J_{24}, and J_{44}: Following the same analysis in the section "Impact", rotate the velocities about $\pi - \phi$, apply γ matrix, then rotate around $-\phi\prime$. The coefficients are then the matrix values.

Putting this together gives:

$$J_I = \begin{pmatrix} \sqrt{\sin^2\phi + \gamma^2\cos^2\phi} & 0 & \tan\phi(1 - \sqrt{\sin^2\phi + \gamma^2\cos^2\phi}) & 0 \\ 0 & \sqrt{\sin^2\phi + \gamma^2\cos^2\phi} & 0 & \frac{(1-\gamma^2)\sin\phi}{\sqrt{\gamma^2+\tan^2\phi}} \\ 0 & 0 & -\frac{\gamma}{\sqrt{\sin^2\phi+\gamma^2\cos^2\phi}} & 0 \\ 0 & 0 & 0 & -\frac{\gamma}{\sqrt{\sin^2\phi+\gamma^2\cos^2\phi}} \end{pmatrix} \tag{7}$$

Surface curvature changes J_{43} and J_{23}.

1.4 Surface Reference Frame Matrix

The natural reference frame that simplifies the matrix considerably is the surface's frame. Coordinates are now defined as variation along the table, and normal to it: $(\delta q_{||}, \delta v_{||}, \delta q_{\perp}, \delta v_{\perp})$. The rotation will then be along the $\hat{t}$ angle, given in the previous section. Now for the matrix J_{22} and J_{44}: The following equations dictate the velocities:

$$\begin{aligned} v_{||}^{+} &= v_{||}^{-} \\ v_{\perp}^{+} &= -\gamma v_{\perp}^{-} \end{aligned}$$

which implies that

$$\begin{aligned} \delta v_{||}^{+} &= \delta v_{||}^{-} \\ \delta v_{\perp}^{+} &= -\gamma \delta v_{\perp}^{-} \end{aligned} \tag{8}$$

J_{13} and J_{33}: Referencing the picture

$$\begin{aligned}
\Delta q\prime &= \frac{\delta q_\perp^-}{\cos\phi} \\
\Delta q\prime\prime &= \Delta q\prime \frac{v^+}{v^-} \\
&= \delta q_\perp^- \sqrt{\gamma^2 + \tan^2(\phi)} \\
\frac{\delta q_\perp^-}{\cos\phi} &= \frac{\Delta_{||}\prime}{\sin\phi} \\
\Rightarrow \Delta_{||}\prime &= \delta q_\perp^- \tan\phi
\end{aligned} \tag{9}$$

given that

$$\begin{aligned}
\delta q_{||}^+ &= \Delta_{||}\prime - \Delta q\prime\prime \sin(\phi\prime) \\
\sin\phi\prime &= \frac{\sin\phi}{\sqrt{\sin^2\phi + \gamma^2\cos^2\phi}} \\
\Rightarrow \delta q_{||}^+ &= \delta q_\perp^- \tan(\phi)(1-1) \\
\therefore \delta q_{||}^+ &= 0 \delta q_\perp^- \delta q_\perp^+ & &= \Delta q\prime\prime \cos(\phi\prime) \\
\therefore \delta q_\perp^+ &= -\gamma \delta q_\perp^-
\end{aligned} \tag{10}$$

So this linear stability matrix is given by:

$$J_I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\gamma & 0 \\ 0 & 0 & 0 & -\gamma \end{pmatrix} \tag{11}$$

References

[1] L. Maťyaś, R. Klages. *Physica D* **187**, 165 (2004).