A force from *nothing* onto *nothing*: Casimir interactions between bubbles in the Fermi sea

Andreas Wirzba

FZ Jülich, Germany

Agenda:

1. General remarks about Casimir effect
2. Geometry dependence
3. Generalization to bubbles in the Fermi sea
4. Interaction of two, three and four bubbles
5. Conclusions

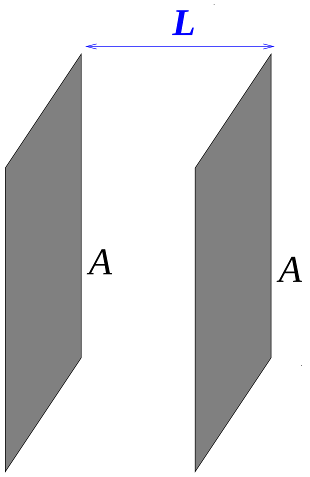
Reference:

Aurel Bulgac & A.W., *Phys. Rev. Lett.* **87** (2001) 120404, nucl-th/0102018

see also:

Physics News Update No. 556#1, Sep. 13, 2001;
Boston Globe, Sep. 25, 2001, p. C2;
Frankfurter Allgemeine Zeitung, Sep. 26, 2001, p. N1.

- H.B.G. Casimir (1948): two parallel uncharged metallic plates attract each other in vacuum



$$\frac{F^{||}(L)}{A} = -\frac{\hbar c}{L^4}\frac{\pi^2}{240} \approx -1.3 \times 10^{-7}\frac{1}{L^4}\mathsf{N}\frac{\mu\mathrm{m}^4}{\mathrm{cm}^2}$$

$$\mathcal{E}^{||}(L) = -\frac{\hbar c}{L^3}\frac{\pi^2}{720}A$$

- Origin: zero-point fluctuations of e.m. field modified by the introduction of the two plates relative to free case $\Rightarrow$ *change* in the energy of the vacuum

> Casimir effect: *Mascroscopic manifestation of quantum fluctuations of the vacuum*

- Related to van der Waals forces (however, the latter always attractive ! Non-trivial geometry-dependence)
- only recently confirmed experimentally (for the sphere-plate system instead of two plates!)
  - S. Lamoureaux, P.R.L. **78** (1997);
  - U. Mohideen & A. Roy, P.R.L. **81** (1998);
  - 9-Feb-2001 issue of New York Times about Casimir effect in *MicroElectroMechanical Systems.*

Casimir effect and vacuum energy in QFT:

- Infinite zero-point energy can be subtracted (discarded) by suitable redefinition of energy-origin (e.g. normal ordering)
- However, energy-origin can be re-defined only *once* (and only in homog. space (no b.c.'s) and without gravity)

$\therefore$ Casmir effect = difference of (properly regularized)

eigen-mode sums of *constrained* QFT

− eigen-mode sums of *free* QFT

$$\varepsilon_C = \lim_{V\to\infty} \frac{\mathcal{E}_C}{V} \qquad (V \text{ : quantization box})$$

$$= \lim_{\Lambda\to\infty} \lim_{V\to\infty} \underbrace{(-1)^{2S}}_{\text{statistic}} \left(\sum\!\!\!\!\!\!\!\int_{\vec{k},\nu_{\rm deg}} \frac{1}{2}\hbar\omega_k|_{V,\Lambda,C} - \sum\!\!\!\!\!\!\!\int_{\vec{k},\nu_{\rm deg}} \frac{1}{2}\hbar\omega_k|_{V,\Lambda,\emptyset} \right)$$

Constraints by
- gravity (e.g. non-flat metric)
- external fields (e.g. anomalies)
- internal fields (e.g. QCD-vacuum)
- geom.-dep. b.c.'s (2 plates, 2 half-spheres, ...)

Gravity serious: “cosmological constant problem!”

The *critical mass density* $\rho_{\mathrm{crit.}} = \dfrac{3H_0^2}{8\pi G_N} \approx h_0^2 \times 10.5\,\mathrm{GeV m}^{-3}$

$$= h_0^2 \times 4.8 \times 10^{-47}\,\mathrm{GeV}^4$$

compared with *vacuum energy density* $\rho_V = \dfrac{\Lambda_{cosm.}}{8\pi G_N} + \varepsilon$

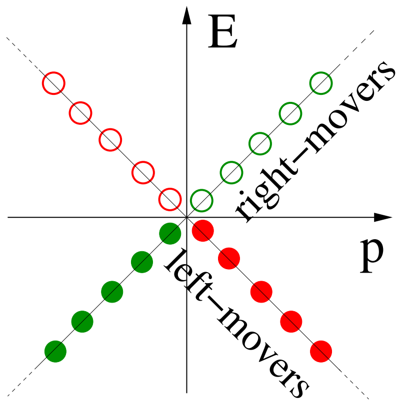
where $\varepsilon = \dfrac{\mathcal{E}_0}{V} = \displaystyle\int \frac{d^3k}{(2\pi)^3}\, \tfrac{1}{2}\hbar\omega_k e^{-\hbar\omega_k/\Lambda_{UV}} \approx \frac{3\hbar c}{2\pi^2}\left(\frac{\Lambda_{UV}}{\hbar c}\right)^4$

Theory	Λ_{UV}	Scale [GeV]	$\varepsilon =$“$\frac{1}{2}\int \hbar\omega$”	$\varepsilon/\rho_{\mathrm{crit.}}$
ToE	M_P	1.2×10^{19}	10^{76} GeV^4	10^{123}
	String	5×10^{17}	10^{72} GeV^4	10^{120}
GUT	SU(5)$_{\mathrm{SUSY}}$	2×10^{16}	10^{65} GeV^4	10^{112}
	SU(5)$_{\mathrm{min.}}$	10^{15}	10^{60} GeV^4	10^{107}
	SUSY$_{\mathrm{break}}$	10^3	10^{12} GeV^4	10^{59}
EW	Higgs vacuum	2.5×10^2	10^9 GeV^4	10^{56}
	Z, W	10^2	10^8 GeV^4	10^{55}
QCD	$\Lambda_{had.}$	1	1GeV^4	10^{47}
	Λ_{QCD}	0.2	10^{-3} GeV^4	10^{44}
EM	$\hbar\omega_{\mathrm{phot.}}$	1 eV	10^{-36} GeV^4	10^{11}
?	10^{-3} eV	$\sim 40K$	10^{-46} GeV^4	1

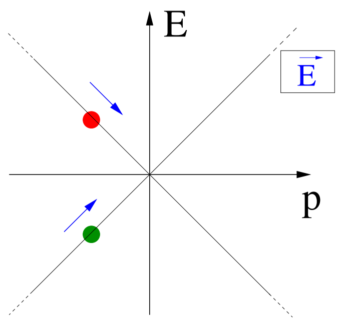
Constraints by external fields

e.g. abelian $U_A(1)$ anomaly

1. Dispersion for massless fermions in 1+1 D.

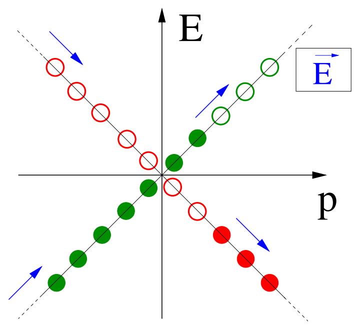


2. Add electric field:



   Nothing happens to single states, except energy change.

3. However, for *infinite mode sums* the level shifts induce



   a surplace of *right-movers* w.r.t. *left-movers*:

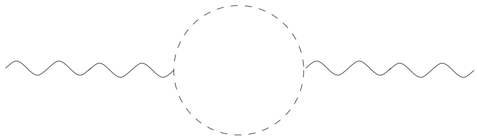
$$\dot{Q}_V = \dot{Q}_R + \dot{Q}_L = 0 \qquad (\text{ vector charge still conserved})$$

$$\dot{Q}_A = \dot{Q}_R - \dot{Q}_L \neq 0 \qquad (\text{ axial charge not conserved})$$

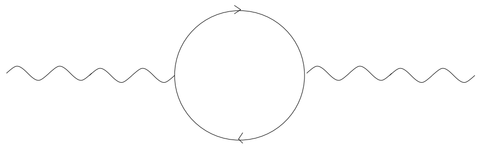
“Strongly interacting” internal fields

Quantum Chromo Dynamics (QCD) generates internal color-magnetic gluon-background fields (*Savvidy* or *Copenhagen* vacuum (1977-80))

- Gluons: Spin 1, *paramagnetic*, and couple to background-field because of gluon self-interactions.



- Quarks: Spin 1/2, however, *diamagnetic* because of Fermi-Dirac statistics.



Nielsen-Hughes-Formula:

Landau levels: $E(p_z, n, S_z) = \left[p_z^2 + 2eB\left(n + \frac{1}{2} - S_z\right)\right]^{1/2}$,

$$(n = 0, 1, 2, \cdots; \quad S_z = \pm S)$$

$$\therefore \varepsilon_{\rm vac} = -\tfrac{1}{2}\chi B^2 = (-1)^{2S}\frac{eB}{2\pi}\int\frac{dp_z}{2\pi}\sum_n\sum_{S_z}E(p_z, n, S_z)$$

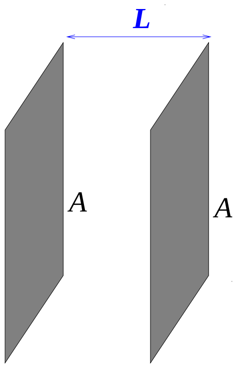
$$= -\frac{1}{2}\left\{(-1)^{2S}\frac{e^2}{8\pi^2}\left[(2S)^2 - \frac{1}{3}\right]\ln\frac{\Lambda_{UV}^2}{CeB}\right\}B^2 + \text{const}$$

where $e = Qg$ and Q are the color-hypercharges.

Thus dielectric constant of the QCD vacuum $\epsilon = 1/\mu < 1$

$\Rightarrow$ *Asymptotic freedom* (from paramagnetic nature of gluons!)

Original Casimir effect:



1) "box" geometry

2) EM b.c.'s : $\mathbf{n}\cdot\mathbf{B}=0$ & $\mathbf{n}\wedge\mathbf{E}=0.$

$\Rightarrow$ 2 scalar fields:

TE: $\phi(\mathbf{x},t)|_{z=0}=\phi(\mathbf{x},t)|_{z=L}=0$ (Dirichlet b.c.'s)

$$\phi^{\mathrm{D}}_{nk_z}(\mathbf{x},\mathbf{t})=\sin(k_z z)\,e^{i\mathbf{k}_T\cdot\mathbf{x}_T}\,e^{-i\omega_{n,\mathbf{k}_T}t}\quad(\mathbf{x}_T=(x,y))$$

$$\boxed{k_z=\left(\tfrac{\pi}{L}\right)n\,,\quad n=1,2,3,\cdots;\quad \omega_{n,\mathbf{k}_T}=c\sqrt{\mathbf{k}_T^2+\left(\tfrac{n\pi}{L}\right)^2}}$$

TM: $\frac{\partial}{\partial z}\phi(\mathbf{x},t)|_{z=0}=\frac{\partial}{\partial z}\phi(\mathbf{x},t)|_{z=L}=0$ (Neumann b.c.'s)

$$\phi^{\mathrm{N}}_{nk_z}(\mathbf{x},t)=\cos(k_z z)\,e^{i\mathbf{k}_T\cdot\mathbf{x}_T}\,e^{-i\omega_{n,\mathbf{k}_T}t}$$

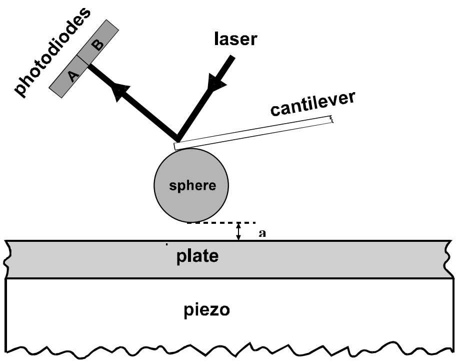
$$\boxed{k_z=\left(\tfrac{\pi}{L}\right)n\,,\quad n=0,1,2,3,\cdots.}$$

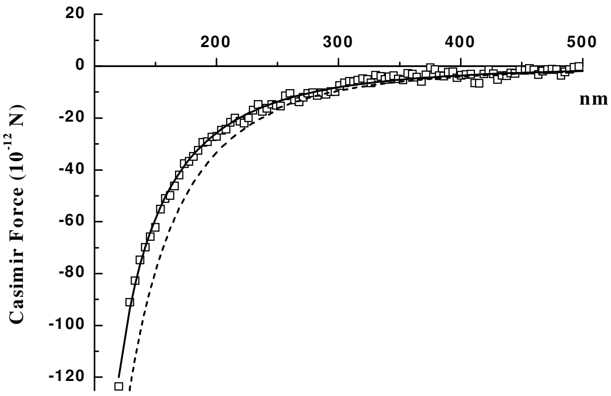
$$\mathcal{E}_C^{||\mathrm{EM}}=\lim_{\Lambda\to\infty}A\int\frac{d^2k}{(2\pi)^2}\left(\sum_{n=1}^{\infty}\tfrac{1}{2}\hbar\omega_{n,\mathbf{k}_T}+\sum_{n=0}^{\infty}\tfrac{1}{2}\hbar\omega_{n,\mathbf{k}_T}\right)e^{-\frac{\hbar\omega_{n,\mathbf{k}_T}}{\Lambda}}$$

$$=\lim_{\Lambda\to\infty}\hbar cAL\left[\frac{3(\Lambda/\hbar c)^4}{\pi^2}-\frac{\pi^2}{720L^4}+\mathcal{O}(\hbar c/\Lambda)\right]$$

Experiments: (< 12 until 1998)

- *Early efforts:* relied on cantilever or spring balances
  - J.T.G. Overbeek & M.J. Sparnaay (1954);
    B.V. Deriagin & I.I. Abrikosova (1957); (glass or quartz !)
    M.J. Sparnaay (1958): first exp. with two *metal* plates
  - sensitivity: (0.1 – 1) $\times 10^{-8}\,\mathrm{N}$ and 17 mV. Outcome: exponent n in the $1/L^n$ Casimir -force determined as 4 ± 1.
  - P.H.G.M. van Blokland & J.T.G. Overbeek (1987)
- *Resurgence: (1996 $\longrightarrow$):* for sphere-plate systems
  under the proximity force theorem: $F^{\circ|}(L) = 2\pi R \frac{-\hbar c \pi^2}{720 L^3}$
  - S.K. Lamoreaux, PRL 78 (1997) & PRL 81 (1998):
    torsion pendulum (60 cm !); mechanical hysteresis eliminated; piezzoelectrical transducers; $\Delta F{=}10^{-11}\,\mathrm{N}$, 10% accuracy
  - U. Mohideen & A. Roy, PRL 81 (1998) and
    A. Roy et al., PRD 60 (1999): atomic force microscope; 32 – 1000 nm, $\Delta F = 10^{-12}\,\mathrm{N}$ and accuracy 1 %
  - H.B. Chan, F. Capasso et al., Science 291 (2001)
    (N.Y. Times, 9-Feb.-2001) capacitance bridge; seesaw device; Microelectromechanical Systems (MEMS); $L_{min} \approx 70\,\mathrm{nm}$
- Ideal case:
  - perfect conductors without impurities at $T = 0$.
- Reality:
  - finite temperature (only important for $L > 1\,\mu\mathrm{m}$); imperfect conductors (realistic metals, Lifshitz theory); surface roughness (geometry averaging, $\langle r^2 \rangle^{1/2}_{\mathrm{s.r.}} \approx$ 3 nm)





see M. Bordag, U. Mohideen, V.M. Mostepanenko, quant-ph/0106045

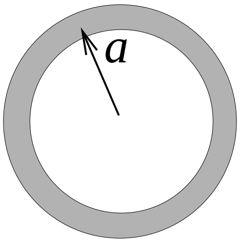
Geometry dependence

- Before 1968:

$$\text{Casimir effect} \leftrightarrow \text{van der Waals forces}$$

$$\left\{\begin{array}{c}\text{2 parallel plates}\\ \text{plate \& sphere}\\ \text{sphere \& sphere}\end{array}\right\} \quad \text{attractive}$$

$\therefore$ Casimir's idea of stabilizing the electron (1953):



electron as spherical shell of radius a with uniform charge distribution.

Then:

$$\mathcal{E}_{em} = \frac{e^2}{2a}$$

$$\mathcal{E}_C \stackrel{?}{=} -C\,\frac{\hbar c}{2a^3}\,4\pi a^2$$

$\Rightarrow$ stabilization if $C = \frac{e^2}{4\pi\hbar c}$ (indep.'ly of radius a).

- However, T. Boyer (1968): force is repulsive!
- (Similarly, R. Balian & C. Bloch (1972-74), Casimir calculations in cavities depend on (bag-)geometry)

$\Rightarrow$ $\boxed{\text{“Generalization” of concept of Casimir energy}}$

“Generalization” of concept of Casimir energy:

1) *geometry dependence*

Casimir energy $\equiv$ vacuum energy from integrating the *geometry-dependent* part of the density of states (d.o.s)

($\leftrightarrow$ *shell* correction energy in Nucl. Phys.)

d.o.s.: $$\rho(E) \equiv \sum_{E_k} \delta(E - E_k)$$

N.o.s.: $$N(E) \equiv \sum_{E_k} \Theta(E - E_k) = \int_{-\infty}^{E} dE' \, \rho(E')$$

Casimir Energy: $$\mathcal{E}_C \equiv \int dE \, E \, \rho_C(E, \textit{geom.-dep.})$$

$$= -\int dE \, N_C(E, \textit{geom.-dep.})$$

Further "Generalization" of concept of Casimir energy:

2) *matter fields*

- *Here*: space not "filled" with *fluctuating* EM modes, but with gas of *non-interacting* (non-relat.) fermions.
- *Similarity* : $\exists$ mode sum $\sum_k \hbar\omega_k$
  with *constant* degeneracy factor,
  because of Pauli's exclusion principle.
- *Difference* : $\exists$ of second scale:
  fermi energy $=$ *chemical potential* μ (at $T \approx 0$)
  in addition to geometric distance scale(s).

Concrete case: Matter filling up the space between voids

How do voids (bubbles) interact with one another under the quantum pressure of a (non-relativistic) background of non-interacting Fermion fields (at zero temperature)?

$\therefore$ effective interaction between empty regions of space ?

Physics realization:

In the crust of a neutron star:

- phenomena as *thermal X-ray emission* from stellar surface, *X-ray burst sources*, *sudden speed-ups* in rotation rate.
- In inner crust, nuclei loose neutrons due to increasing pressure and density: Hence below saturation density, $\exists$ nuclei of exotic shapes immersed in a neutron gas:

  $\Rightarrow$ chain of phases with increasing ρ (0.03 – 0.1 fm^{-3}):

  nuclei $\rightarrow$ rods $\rightarrow$ slabs $\rightarrow$ tubes $\rightarrow$ bubbles $\rightarrow$ uniform matter

meat-ball, spaghetti, lasagne vacua from
interplay between Coulomb and surface energies
with phase differences of a few keV/fm^3.
Liquid drop model: shell correction energy neglected !
(see A. Bulgac & P. Magierski, Nucl. Phys. A 683 (2001) 695.)

Inside a neutron/quark star:

- Quark matter droplets immersed in hadron matter at several times nuclear matter densities.

In the lab:

- Buckyballs immersed in an electron gas (in liquid mercury)
- or Buckyballs immersed in liquid ^{3}He.
- (Bosonic) cavities in dilute atomic Fermi condensates.

Calculation

- Casimir energy for fermions between two impenetrable (parallel) planes of distance L:

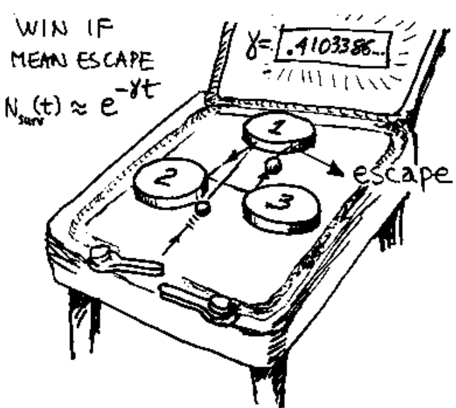
$$\mathcal{E}_C = \mu\, F(k_F L) \qquad \text{with } \mu = \hbar^2 k_F^2/2m \text{ natural scale .}$$

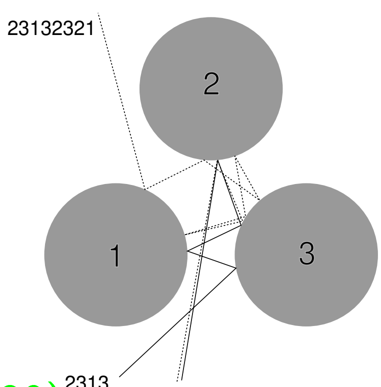
- More complicated geometries $\rightarrow$ involved computations.
- Case of embedded non-overlapping *n-sphere* or n-rod systems (cylinders or disks in 2D) still relatively simple:
  - *Krein trace-formula* relates M.G. Krein (1953, 1962), Birman (1962)
    change in level density $\delta\rho(E)$ $\leftrightarrow$ *change* in phase shift
    ($:= \ln\det S_n = \mathrm{tr}\ln S_n$ of the n-sphere S-matrix)

$$\boxed{\delta\rho(E) = \bar{\rho}(E) - \bar{\rho}_0(E) = \frac{1}{2\pi i}\,\frac{d}{dE}\mathrm{tr}\ln S_n(E).}$$

  - Then extract *geometry-dependent* Casimir fluctuations from the *multiple*-scattering part

    $\Rightarrow$ Calculation mapped to QM (& semiclassics) of hyperbolic n-sphere & n-disk scattering problems





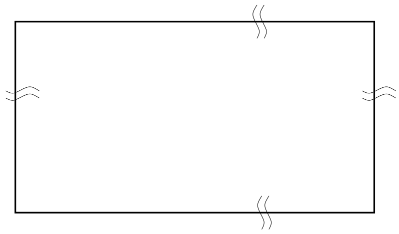
References:

P. Gaspard & S. Rice, *J. Chem. Phys.* 90 (1989)

M. Henseler, A. Wirzba & T. Guhr, *Ann. Phys.* 258 (1997)

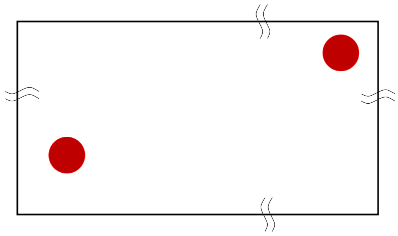
1. *no bubbles:*

$$\rho(E) = \rho_0(E) \quad \text{(background)}$$



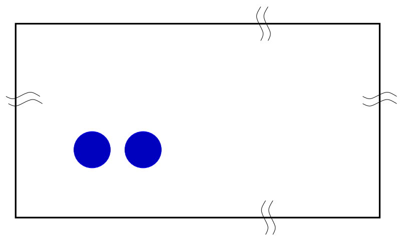
2. n bubbles (of radii a_i) added incoherently or "infinitely" far separated (note the excluded volume):

$$\rho(E) = \rho_0(E) + \underbrace{\sum_{i=1}^{n} \rho_W(E, a_i)}_{\textit{Weyl term}}$$



3. geometry-dependent arrangement of n bubbles

$$\rho(E) = \rho_0(E) + \sum_{i=1}^{n} \rho_W(E, a_i) + \rho_C(E, \{a_i\}, \{\mathbf{r}_{ij}\})$$



4. Krein formula (note the averaging):

$$\delta\rho(E) = \bar{\rho}(E) - \bar{\rho}_0(E) = \frac{1}{2\pi i}\frac{d}{dE}\ln \underbrace{\det S_n(E)}_{e^{2i\eta_n(E)}}$$

5. *Multiple scattering matrix*: (see A.W., Phys. Rep. 309 ('99))

$$\therefore \quad \bar{\rho}_C(E, \{a_i\}, \{\mathbf{r}_{ij}\}) = -\frac{1}{\pi}\mathrm{Im}\left(\frac{d}{dE}\ln\det M(E)\right)$$

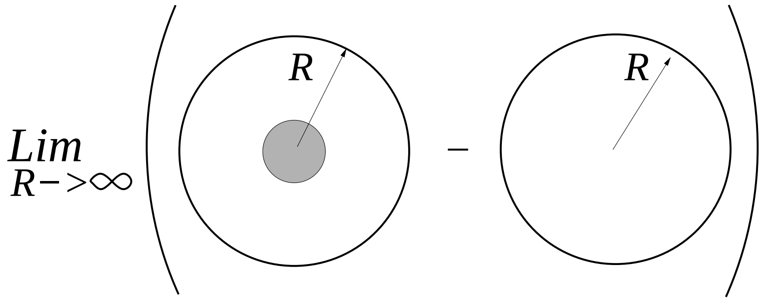
6. Casimir energy:

$$\mathcal{E}_C = \int_0^{\mu} dE\,(E-\mu)\,\bar{\rho}_C = -\int_0^{\mu} dE\,\overline{\mathcal{N}}_C$$

Digression 1: *E.Beth & G.E. Uhlenbeck (1937)*

predecessor of *Krein (1962)* formula for *spherical* potentials:

Idea: *spherical boxes:*



Asymptotically:

$$u_{k\ell}(r) \sim \sin\left(kr - \tfrac{1}{2}\,\ell\pi + \eta_\ell(k)\right)$$

$$u^{(0)}_{k\ell}(r) \sim \sin\left(kr - \tfrac{1}{2}\ell\pi\right)$$

and Dirichlet b.c.'s: $\quad u_{k\ell}(R) = u^{(0)}_{k\ell}(R) = 0.$

$\Rightarrow$

EV conditions ($(2\ell+1)$-fold degenerate) :

$$k_{\ell,n}R - \tfrac{1}{2}\ell\pi + \eta_\ell(k_{\ell,n}) = \pi n \quad \text{(with potential)}$$

$$k^{(0)}_{\ell,n}R - \tfrac{1}{2}\ell\pi = \pi n \quad \text{(without potential)}$$

$$n = 0, 1, 2, \cdots$$

A change of n by one unit, for ℓ fixed, leads to

$$\Delta k_\ell \left(R + \frac{\partial}{\partial k}\eta_\ell(k) \right) = \pi = \Delta k^{(0)}_\ell \, R \, ,$$

such that the conditions

$$\bar\rho_\ell(k)\Delta k_\ell = \bar\rho^{(0)}_\ell(k)\Delta k^{(0)}_\ell = 2\ell + 1 \quad \text{(note the averaging !)}$$

induce the formula $\boxed{\bar\rho_\ell(k) - \bar\rho^{(0)}_\ell(k) = \frac{2\ell+1}{\pi}\frac{\partial}{\partial k}\eta_\ell(k)}$

Digression 2 : Extraction of multi-scattering matrix

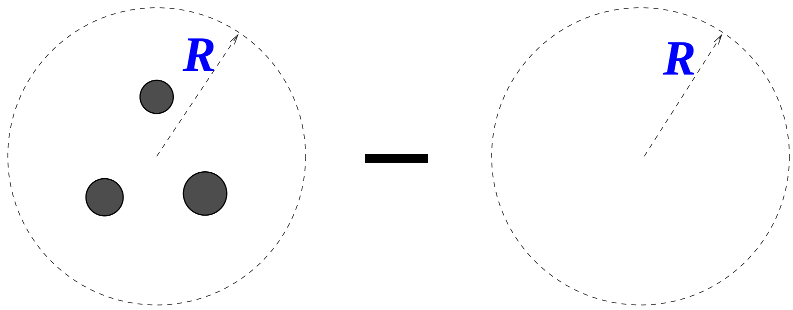
$$
\begin{aligned}
\delta\rho(E) &= \bar\rho(E) - \bar\rho_0(E) \\
&= \sum_{i=1}^{n} \bar\rho_W(E, a_i) + \bar\rho_C(E, \{a_i\}, \{\mathbf{r}_{ij}\}) \\
&= \frac{1}{\pi}\frac{d}{dE}\eta_n(E) \quad (\text{total phase shift}) \\
&= \frac{1}{2\pi i}\frac{d}{dE} \ln \overbrace{\det S_n(E)}^{e^{2i\eta_n(E)}} \quad (\text{i.g. non-spherical geometry}) \\
&= \frac{1}{2\pi i}\frac{d}{dE} \ln \left\{ \left(\prod_{i=1}^{n} \underbrace{\det S_1(E, a_i)}_{\textit{single sphere}} \right) \underbrace{\frac{\det M^\dagger(k^*)}{\det M(k)}}_{\textit{multi-scat.}} \right\} \\
&= \sum_{i=1}^{n} \underbrace{\frac{1}{\pi}\frac{d}{dE}\eta_1(E, a_i)}_{\bar\rho_W(E, a_i)} + \frac{1}{2\pi i}\frac{d}{dE}\left(\ln\det M^\dagger - \ln\det M\right)
\end{aligned}
$$

$$
\therefore \quad \bar\rho_C(E, \{a_i\}, \{\mathbf{r}_{ij}\}) = \underbrace{-\frac{1}{\pi}\mathrm{Im}\left(\frac{d}{dE}\ln\det M(E)\right)}_{\text{semiclass'ly: periodic orbit sum}}
$$

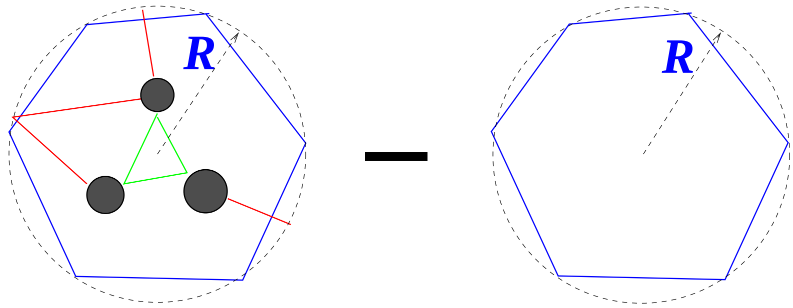
(see A.W., Phys. Rep. 309 (1999))

Digression 3: *Semiclassical interpretation of Krein formula*

Consider the difference of the densities of states of two *bounded reference systems* :



$\exists$ *spurious orbits* whose length grows with increasing R:



Removal of long orbits by exponential damping or averaging:

$$\lim_{\epsilon\to 0_+} \lim_{R\to\infty} \left\{ \rho^{(n)}(k+i\epsilon; R) - \rho^{(0)}(k+i\epsilon; R) \right\}$$

$$= \frac{1}{2\pi} \mathrm{Im} \frac{\mathrm{d}}{\mathrm{d}k} \ln \det \mathbf{S}^{(n)}(k) \Big|_{k\,\mathrm{real}}$$

Digression 4: Casimir energy in fermionic background

chemical potential $\mu = \hbar^2 k_F^2/2m$ or Fermi momentum k_F provide for a *natural UV-cutoff* $(\Lambda_{UV} := \mu$ or $k_{UV} := k_F)$

Casimir energy at *fixed* total number of fermions $\mathcal{N}$:

$\Rightarrow$ two different chemical potentials μ and μ_0 at *finite* and *infinite* separation of the scatterers, respectively.

$$\mathcal{N} = \int_{-\infty}^{\mu} \rho(E)\, dE = \int_{-\infty}^{\mu_0} [\rho_0(E) + \rho_W(E)]\, dE$$

$$\Rightarrow \quad \int_{\mu_0}^{\mu} \rho(E)\, dE = -\int_0^{\mu_0} \underbrace{[\rho(E) - \rho_0(E) - \rho_W(E)]}_{\rho_C(E) = \frac{d}{dE}\mathcal{N}_C(E)} dE$$

$$\boxed{\mathcal{E}_C|_{\mathcal{N}}} = \int_{-\infty}^{\mu} E\rho(E)\, dE - \int_{-\infty}^{\mu_0} E[\rho_0(E) + \rho_W(E)]\, dE$$

$$= \int_{-\infty}^{\mu_0} E\, \rho_C(E)\, dE + \int_{\mu_0}^{\mu} \mu_0\, \rho(E)\, dE + \mathcal{O}(V^{-1})$$

$$= \int_{-\infty}^{\mu_0} (E - \mu_0)\, \rho_C(E)\, dE = \boxed{-\int_{-\infty}^{\mu_0} \mathcal{N}_C(E) dE\,,}$$

Grandcanonical Casimir energy at *fixed* $\mu = \mu_0$:

$$\widetilde{\mathcal{E}_C}|_{\mu} - \mu \mathcal{N}_C(\mu) = \mathcal{E}_C|_{\mathcal{N}} + \mathcal{O}(V^{-1})$$

Digression 5: *Multi-scattering matrix for n spheres of radii a_j and relative distance $r_{jj'}$ with $1 \leq j, j' \leq n$:*

$$M^{jj'}_{lm,l'm'} = \delta^{jj'}\delta_{ll'}\delta_{mm'} + (1-\delta^{jj'})\left(\frac{a_j}{a_{j'}}\right)^2 \sqrt{4\pi}(-1)^{m+l'}$$
$$\times \frac{j_l(ka_j)}{h^{(1)}_{l'}(ka_{j'})} \sqrt{(2l+1)(2l'+1)}\, i^{-l'-l}$$
$$\times \sum_{\tilde{l}=0}^{\infty} h^{(1)}_{\tilde{l}}(kr_{jj'}) \sqrt{2\tilde{l}+1}\, i^{\tilde{l}} \begin{pmatrix} \tilde{l} & l' & l \\ 0 & 0 & 0 \end{pmatrix}$$
$$\times \sum_{\tilde{m}=-l'}^{l'} \begin{pmatrix} \tilde{l} & l' & l \\ m-\tilde{m} & \tilde{m} & -m \end{pmatrix} Y_{\tilde{l},m-\tilde{m}}\left(\hat{r}^{(j)}_{jj'}\right) D^{l'}_{m',\tilde{m}}(j,j')$$

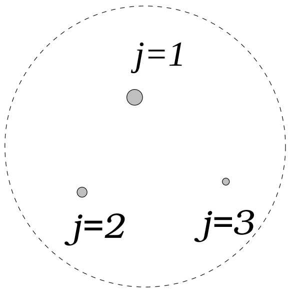
Multi-scattering matrix for 2-spheres of radii a and distance r :

$$M^{(\pm)}_{lm,l'm'} = \delta_{ll'}\delta_{mm'}$$
$$\pm\, \delta_{mm'} \frac{j_l(ka)}{h^{(1)}_{l'}(ka)} \sqrt{(2l+1)(2l'+1)}\, i^{-l'-l}$$
$$\times \underbrace{\sum_{\tilde{l}=0}^{\infty} h^{(1)}_{\tilde{l}}(kr)\, (2\tilde{l}+1)\, i^{\tilde{l}} \begin{pmatrix} \tilde{l} & l' & l \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{l} & l' & l \\ 0 & m & -m \end{pmatrix}}_{\approx\ \delta_{m0}\frac{\exp(ikr)}{ikr} \quad \text{for } kr \gg 1}$$

M. Henseler, A. Wirzba & T. Guhr, *Ann. Phys.* **258** (1997) 286.

Limit of small scatterers:

$\lambda = 2\pi/k \gg a_i$ for all radii a_i and no overlaps:



$$[M(E)]^{jj'} \approx \delta^{jj'} - \left(1 - \delta^{jj'}\right) \underbrace{f_j(E)}_{\substack{s\text{-wave}\\ \text{ampl.}}} \frac{\exp(ikr_{jj'})}{r_{jj'}}$$

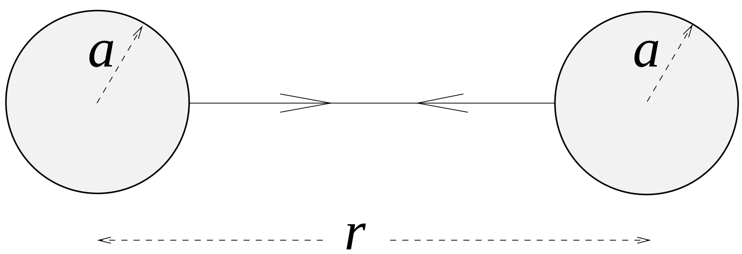
Next order correction from p-wave scattering.

Two spheres of radius a at distance r in the *small-scatterer* limit:

$$\mathcal{N}_C^{\mathrm{oo}}(E) = \nu_{\mathrm{deg}} \frac{a^2}{\pi r^2} \sin[2k(r-a)] + \mathcal{O}\left((ka)^3\right)$$

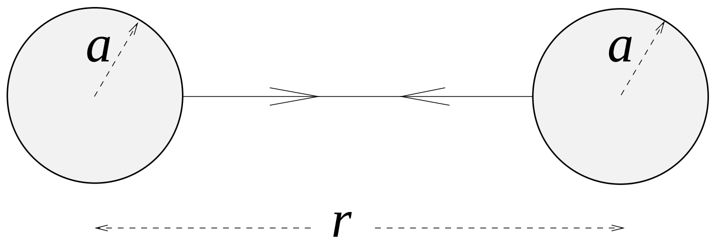
This should be compared with the *semiclassical* result:

$$\mathcal{N}_{C,scl}^{\mathrm{oo}}(E) = \nu_{\mathrm{deg}} \frac{a^2}{4\pi r(r-2a)} \sin[2k(r-2a)]$$



(Gutzwiller's trace formula)

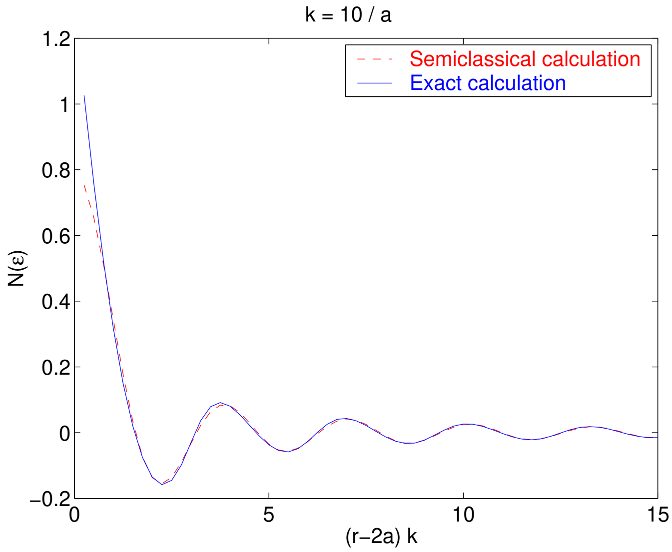
Two-sphere geometry:

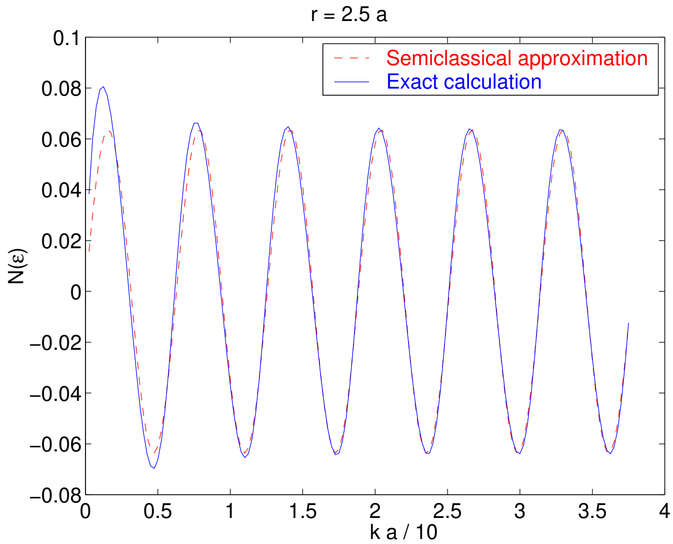


single po, no repetitons;

$S_{po}/\hbar = 2k(r-2a) > 1$:

$$\mathcal{N}_C^{\mathrm{oo}}(E) \approx \nu_{\mathrm{deg}} \frac{a^2}{4\pi r(r-2a)} \sin[2k(r-2a)]$$





$$\mathcal{E}_C^{\mathrm{oo}} = -\int_0^{\mu(k_F)} dE\, \mathcal{N}_C^{oo}(E)$$

$$\approx -\nu_{\mathrm{deg}}\, \mu \frac{a^2}{2\pi r(r-2a)} j_1[2k_F(r-2a)], \qquad (k_F a > 1)$$

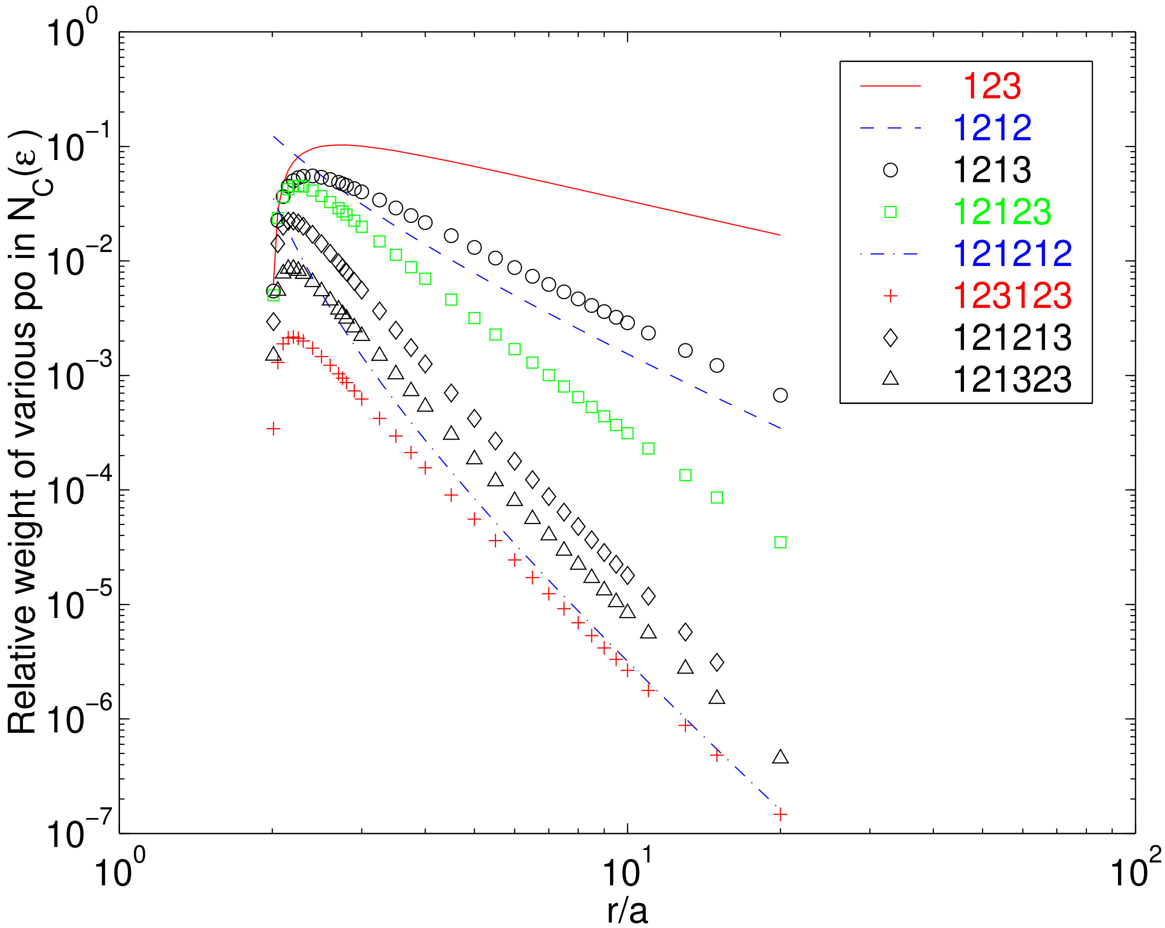
(of longer range than molecular van der Waals forces !)

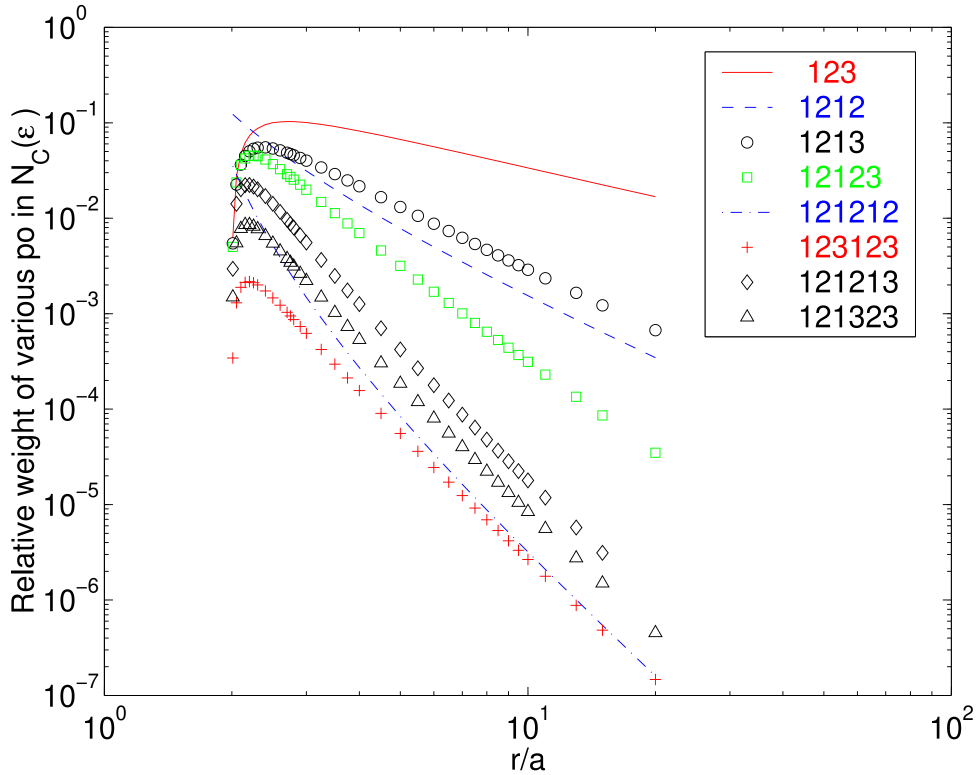
Sphere-plate geometry:

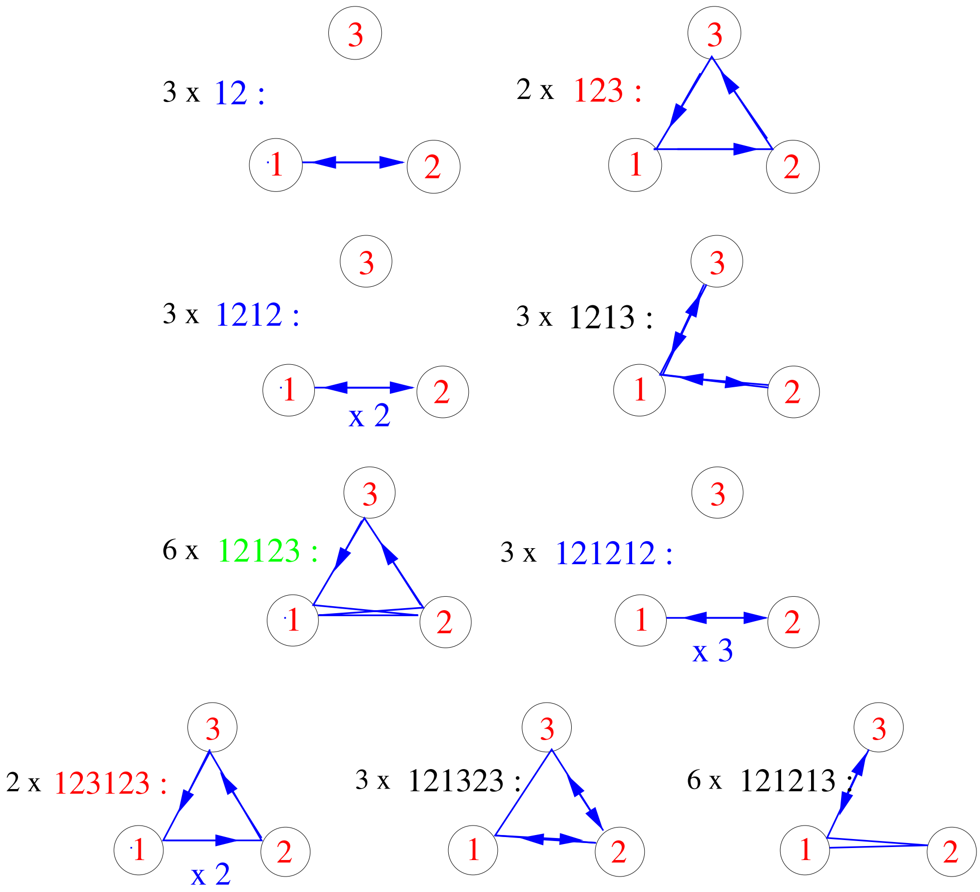
$$\mathcal{E}_C^{\mathrm{o}|} \approx -\nu_{\mathrm{deg}}\, \mu \frac{a}{2\pi(r-a)} j_1[2k_F(r-a)], \qquad (k_F a > 1)$$

Three and four spheres:

- Expectation from semiclassical po summation:
  Casimir energy cannot be represented as sum of pairwise interactions;
  i.e. $\exists$ genuine *three* and *more-body* interactions
- However, relative weights of various po's in $\mathcal{N}_C(E)$ for a system of three identical spheres at the vertices of an equilateral triangle show that 2-bounce orbit dominates (max. correction due to 3-bounce orbit is $\sim 10\,\%$ at $r \approx 2.5a$)







Billiard analogy : difficult to make *long* shots, especially with many bounces – the slightest error ruins the shot.

$k_F a \gg 1$: the Casimir energy satisfies the rules

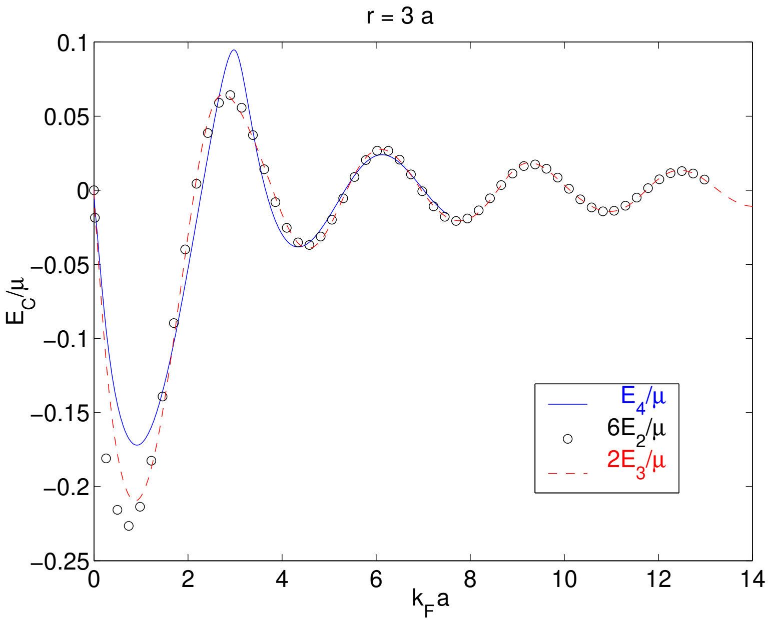
$$E_3 \approx 3E_2$$

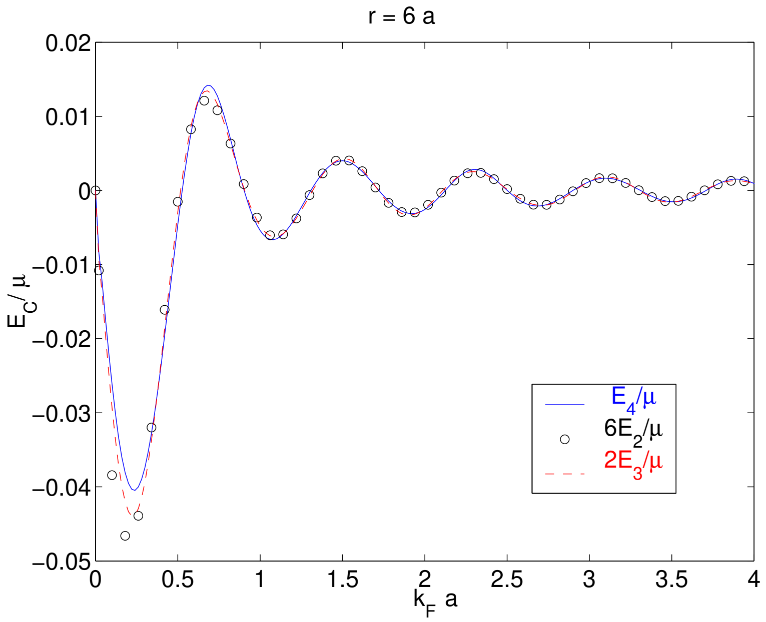
for three identical spheres (equilateral triangle) and

$$E_4 \approx 6E_2 \approx 2E_3$$

for four identical spheres (symmetric tetrahedron):

($k_F a \leq 1$: corrections up to 10 % and 25 % for 3 & 4 spheres)





Conclusions:

- $\exists$ effective interaction between empty regions of space that are embedded in a non-interacting fermion background. New form of Casimir energy: neither attractive or repulsive, but changes according to relative arrangement of scatterers.
- $\exists$ many-body Casimir contributions in n-sphere systems: – Short-ranged and dominated by long-ranged two-body interactions (even for small separations or small scatterers).
- Casimir contributions well approximated semiclassically.
- Presented results for symmetric arrangements of spheres. However, tests of various asymmetrical configurations of three and four spheres show the same general pattern: d.o.s. $\approx$ *sum* of corresponding pairwise contributions.
- The spheres can be replaced with other objects, with curvature radii larger than the Fermi wave length. Effects of finite surface thickness can booked as Weyl-term contributions if the objects do not overlap.
- Nuclei immersed in neutron gas in a neutron star crust: Casimir interactions of same order ($\sim$ keV/fm^3) as energy differences between various liquid drop phases. Long-range and oscillating: $\rightarrow$ disordered lattices expected. Enhancement of disorder: finite T and corrugated surfaces. Pairing interaction: modification of Casimir contributions. (Andreev reflection: focussing and defocussing scattering).
- Most promising case: Buckyball-lattices in liquid mercury.