Lyapunov Exponents for the Stadium Billiard

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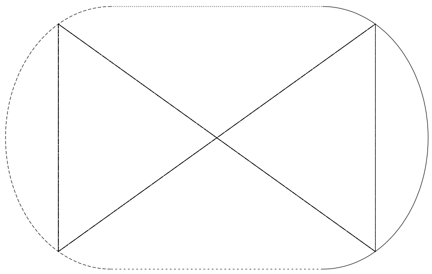

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1 Introduction

This project deals with the 2d-billiard, the **Bunimovich Stadium**. A 2d-billiard is a plane in which the boundary of some figures are placed. A particle then moves around on the plane, eventually bouncing elastically on the boundary of the figures, and eventually escaping the plane.

1.1 The Bunimovich Stadium

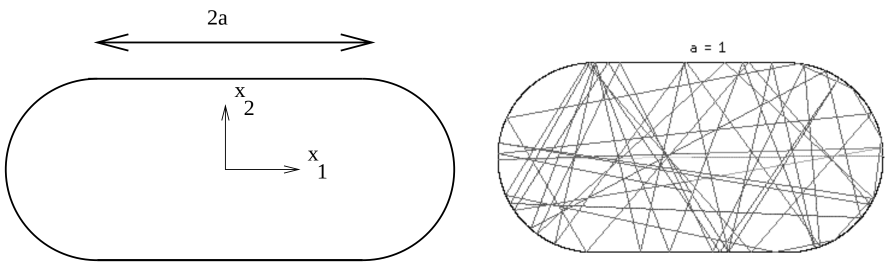

Figure 1: The Bunimovich stadium. And a trajectory in the a=1 stadium.

The bunimovich stadium is a billiard boundary consisting of two radius 1 half-circles joined together by two parallel lines, each of length $2a$. The stadium has 4 points, at which the curvature of the boundary changes discontinuously. These points are the only singular points of the stadium system and are the points where the half-circles connects the stadight lines. We will here look at a dimensionless particle with speed=1 and mass=1 rolling around in this billiard. We assume all bounces to be elastic and free movement of the particle when it is not hitting the boundary. The particle off course never escapes the stadium. It should be noticed that the stadium consists of two figures, a rectangle and a circle. In each of these figures, the described motion of a particle is integrable, but in the stadium the motion of a particle shows to be chaotic (i.e. non-integrable). It is the mixture of the two each integrable cases that ensures the chaotic behaviour. This phenomenon is called intermittency.

2 Lyapunov Exponent

2.1 Definition

One way to examine the chaotic behaviour of a system is by looking at the lyapunov exponent. The Lyapunov exponent characterises the time separation of originally close phase space orbits. If the original separation is $|\delta\mathbf{x}(0)|$ and the separation to time t is $|\delta\mathbf{x}(t)|$, then if the function

$$|\delta\mathbf{x}(t)| = e^{\lambda t} \cdot |\delta\mathbf{x}(0)| \tag{1}$$

holds in the limit $t \to \infty$, the Lyapunov exponent is defined as λ. The system off course has to be ergodic, otherwise the calculated exponent will not a characteristic of the whole system, and the time has to be so long, that the orbit is dense in the reachable area of phase space, otherwise there could be an attracting fix point or cycle, that would yield a zero lyapunov exponent. An example of such a system is the Henon mapping [1] which is chaotic[2] for some parameters[3] while for other [4] it has a stable period 13-orbit which is first discovered after some 1000 iterations of the map. In the infinitesimal limit of $\delta\mathbf{x}(0)$, the ratio $\frac{\delta\mathbf{x}(t)}{\delta\mathbf{x}(0)}$ is known, as this is the Jacobian matrix $\mathbf{J}$ of the orbit. For large t, the most stable eigendirection of $\mathbf{J}$ wins out with eigenvalue Λ^t, and the Lyapunov exponent is given as

$$\bar{\lambda} = \lim_{t\to\infty} \frac{1}{t} log|\Lambda^t| \tag{2}$$

2.2 For the Stadium Billiard

2.2.1 Theory

For a system as the stadium billiard it is nice to know for what value of a the system becomes most chaotic. One way to do this is simply by finding the Lyapunov Exponent for different values of a and then look for a maximum. The function $(a, \bar{\lambda(a)})$ must clearly have a maximum somewhere, since the stadium is integrable in both limits of a (0 and ∞). For $a = 0$ the billiard is simply the circle, while for $a \to \infty$, the system becomes a rectangle. So $\overline{\lambda(a)} \to 0$ for $a \to 0$ and $\overline{\lambda(a)} \to 0$ for $a \to \infty$. The Jacobian $\mathbf{J}$ for a trajectory in the stadium billiard consists of free flight and rotation parts. For the free flight, $\mathbf{J}$ is

$$\mathbf{J}_s = \begin{vmatrix} 1 & \tau \\ 0 & 1 \end{vmatrix} \tag{3}$$

where τ is the time of flight of the free flight. Bounces on the straight lines multiplies a factor -1 per bounce, which we will omit here, as we are looking at the unfolded stadium (the washboard). For a single bounce in a semi circle, $\mathbf{J}$ is

$$\mathbf{J}_r = \begin{vmatrix} -1 & 0 \\ 2/\cos(\phi) & -1 \end{vmatrix} \tag{4}$$

As the angle ϕ is the same for all bounces in one rotation in a semi circle, the Jacobian for a k-bounce rotation is given as the alternating product of Jacobians for the $k-1$ free flights and the k bounces:

$$\mathbf{J}_{rs} = \mathbf{J}_r \cdot \prod_{j=1}^{k-1} \mathbf{J}_s \cdot \mathbf{J}_r \tag{5}$$

[1] $(x_{n+1}, y_{n+1}) = (a - x_n^2 + by_n, x_n)$

[2] Actually it isn't proved that the orbit is just not a *very* long cycle.

[3] a=1.4, b=0.3

[4] eg. a=1.39945219,b=0.3

Simple geometry gives the time of flight between two bounces in a rotation as $2\cos(\phi)$. $\mathbf{J}_{rs}$ is then calculated as

$$\mathbf{J}_{rs} = \begin{vmatrix} -1 & 0 \\ 2/\cos(\phi) & -1 \end{vmatrix} \cdot \prod_{j=1}^{k-1} \begin{vmatrix} 1 & 2\cos(\phi) \\ 0 & 1 \end{vmatrix} \quad \begin{vmatrix} -1 & 2/\cos(\phi) \\ 0 & -1 \end{vmatrix} \tag{6}$$

to give

$$\mathbf{J}_{rs} = (-1)^k \begin{vmatrix} 1-2k & 2(k-1)\cos(\phi) \\ 2k/\cos(\phi) & 1-2k \end{vmatrix} \tag{7}$$

[5]

2.2.2 Results

The above equation for the Jacobian was used in a computer program with the stadium simulator to calculate the wanted $(a, \bar{\lambda(a)})$ function. The results from these calculations are seen in 2. The maximum for the lyapunov exponent is reached at $a = 0.5$+/-0.01

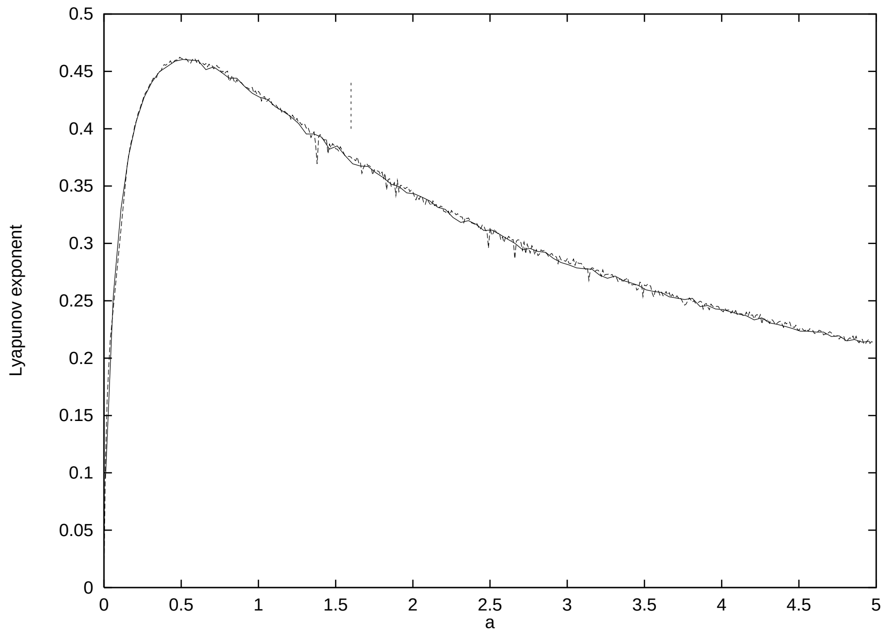

Figure 2: The Lyapunov exponent for the stadium as a function of a. For each a, the value is an average of 10^3 different trajectories, each with $2 \cdot 10^3$ bounces in the half stadium.

[5]This is not the same expression for $\mathbf{J}_{rs}$ as in [Cv95a], but this is the right one, since the expression in [Cv95a] for a k-bounce rotation is a product of k $\mathbf{J}_r$ and k (NOT K-1!) $\mathbf{J}_s$. i.e. $\mathbf{J}_{rs} = \prod_{j=1}^{k} \mathbf{J}_s \cdot \mathbf{J}_r$

3 Periodic Orbit Theory

3.1 Periodic orbits

A periodic orbit in phase space is an orbit that returns to its starting point after its period T_p. The idea of the periodic theory is to use the fundamental local $\leftrightarrow$ global duality: **The spectrum of eigenvalues is dual to the spectrum of periodic geodesics.**

The periodic orbit theory can be used to calculate different averages of a chaotic dynamical system. And macroscopically measurable physical quantities of a chaotic system is exactly such averages. The way to go is by introducing evolution operators, which describes the evolution of densities instead of single particles in phase space. And given an operator, one can find the eigenvalues which are characteristics of it. The spectrum of the evolution operators can be found by using zeta functions or Fredholm determinants. Zeta functions and Fredholm determinants can be found by summing over abilities (i.e. stabilities and times) of periodic orbits in the system. So one has to collect periodic orbits in order to calculate averages in a nice way.

3.2 Symbolic Dynamics

To find periodic orbits in a systematic way, one has to use a *symbolic dynamics*, which is a way of describing a trajectory in phase space by its topological behaviour. It could in the stadium be by describing what lines/arcs it hits in what order [Bih92].

3.2.1 Pruning

When a symbolic dynamics is constructed one has to leave out symbol sequences that has no corresponding orbit. This is called pruning. There are two sorts of pruning rules [Bih92].

- *Geometric* pruning rules, which takes care of orbits that are never realisable, no matter the value of the systems parameter. An example in the stadium for the above mentioned symbolic dynamics is the impossibility for a particle to have two consecutive hits on the same straight line.

- *Dynamical* pruning rules, which takes care of orbits that are admissible for some values of the systems parameter, while prohibited for other. An example in the stadium is the impossibility for an orbit in a very short stadium (small a) to first bounce a lot of times between the straight lines and then have a lot of bounces in an arc.

4 Symbolic Dynamics for the Stadium

In order to apply the periodic theory to the Bunimovich Stadium, one has to define a proper way to do this. Several different symbolic dynamics has been proposed in the literature, [Bih92],[Aki94] and [Cv95a]. [Bih92] uses a symbolic dynamics with one symbol for each of the two sides in the stadium and two symbols for each of the half-cirkels: one symbol for ration clockwise and another for anti-clockwise rotation. this make their alphabet a $2+2\cdot 2 = 6$ letter alphabet. They have to account for both geometric and dynamical pruning rules. One simple geometric pruning rule is e.g. the prohibition of consecutive bounces on the same straight line. [Aki94] uses an 18-letter alphabet. [Cv95a] uses a ∞-letter symbolic dynamics, but takes advantage of the C_{2v}- symmetry of the stadium and of the integrability of the two limits of the stadium parameter, the circle and the rectangle. They use the unfolded stadium, which is constructed by replacing the straight lines with an infinite tile of stadiums without lines, a billiard which boundary resembles an oldfashioned washboard. Their symbolic dynamics has no geometric pruning rules, since all symbol sequences are realisable in the infinitely long stadium. The only pruning left are The calculations here are done with this symbolic dynamics in the stadium fundamental domain (1/4-stadium). To get to the symbolic dynamics in the fundamental domain, we first define it in the half stadium, the stadium where the σ_2 symmetry across the x_2 axis is used: One symbol is defined for the i-th rotation in the half-circle:

$$R_i = \begin{cases} \rho_1, \rho_2, \rho_3, \cdots & \text{rotate clockwise 1, 2, 3... steps} \\ \rho_0 & \text{reflect out in one bounce} \\ \rho_{-1}, \rho_{-2}, \rho_{-3}, \cdots & \text{rotate anti-clockwise 1, 2, 3... steps} \end{cases} \tag{8}$$

and another is defined for the i-th free flight:

$$S_i = \begin{cases} \alpha_1, \alpha_2, \alpha_3, \cdots & \text{ascend the } x_2\text{-axis} \\ \alpha_0 & \text{remain in the tile} \\ \alpha_{-1}, \alpha_{-2}, \alpha_{-3}, \cdots & \text{descend the } x_2\text{-axis} \end{cases} \tag{9}$$

Since a rotation in a half-circle always is followed by a free flight, an itenary for a trajectory consists of pairs R_iS_i, as $..S_{-1}R_{-1}S_0R_0 \cdot S_1R_1S_2R_2...$, where the dots denotes present time. The correspondence between the half- and the full stadium is a reflection σ_2, per translation α_i.

The symbolic dynamics in the fundamental domain is then defined as follows (we here choose the upper right quart-stadium image of the fundamental domain). The fundamental domain takes advantage of the σ_1 reflection symmetry. A trajectory in the fundamental domain leaves the quart-circle, and ascends the x_2 axis, either directly or after a σ_1 reflection. The direction of the next rotation depends on whether or not the trajectory enters the next half-circle above the x_1-axis in the full stadium. In the fundamental domain, this reduces to the number of bounces on the x_1 axis, whether or not there has been an odd or even number of bounces since last leaving of the circle. So the fundamental domain symbolic dynamics is:

$$R_i = \begin{cases} \rho_1, \rho_2, \rho_3, \cdots & \text{rotate 1, 2, 3... steps} \\ \rho_0 & \text{reflect out in one bounce} \\ \rho_{\underline{1}}, \rho_{\underline{2}}, \rho_{\underline{3}}, \cdots & \sigma_1\text{, then rotate 1, 2, 3... steps} \end{cases} \tag{10}$$

$$S_i = \begin{cases} \alpha_1, \alpha_2, \alpha_3, \cdots & \text{ascend the } x_2\text{-axis} \\ \alpha_0 & \text{remain in the tile} \\ \alpha_{\underline{1}}, \alpha_{\underline{2}}, \alpha_{\underline{3}}, \cdots & \sigma_1\text{, then ascend the } x_2\text{-axis} \end{cases} \tag{11}$$

The problem with this symbolic dynamics is, that it is infinite, each symbol can have infinitely many values. And there can be infinitely many symbols. So I have chosen do define cycle length as number of bounces in the fundamental domain (and leave out bounces on the x_1 and x_2 axis).

4.1 Poincare sections.

In each of figure 3 and figure 4, two Poincare sections of a long trajectory can be seen. Figure 3 is for a=10 and figure 4 is for a=1. The left Poincare section is for flight leaving the straight line and the right Poincare section is for flight leaving the arcs. The coordinates used are the position q of the particle along the x_2 axis and sine to the angle ϕ between the normal to the x_2 axis and the velocity of the particle. The Poincare sections are colorcoded after how many bounces/tiles passings the particle have done since last passing the Poincare section line.

4.2 Manifolds

It is interesting to look at the stable/unstable manifolds of a given orbit, and see how they evolve in the Poincare plane. This shows how a little displacement from a given periodic orbit behaves through time. In figure 5, some of the unstable manifolds around the horizontal $\overline{\rho_0\alpha_0}$ are shown. It is seen that they bend at some points. A bending of a manifold takes place whenever the manifold hits one of the singular points on the boundary. One should distinguish between bendings and images of bendings: if a manifold once are bended, it is also bended at the next Poincare section, even though it hasn't crossed a singular point on the boundary. Recently, K. Hansen ([Han95]) has proposed a method of finding periodic orbits by looking at the crossing of the stable and unstable manifolds. The manifolds shown here are the unstable manifolds. The stable manifolds in this case is simply a mirroring through the $\sin(\phi) = 0$ axis in the Poincare section plane, since our stadium has time reversal symmetry.

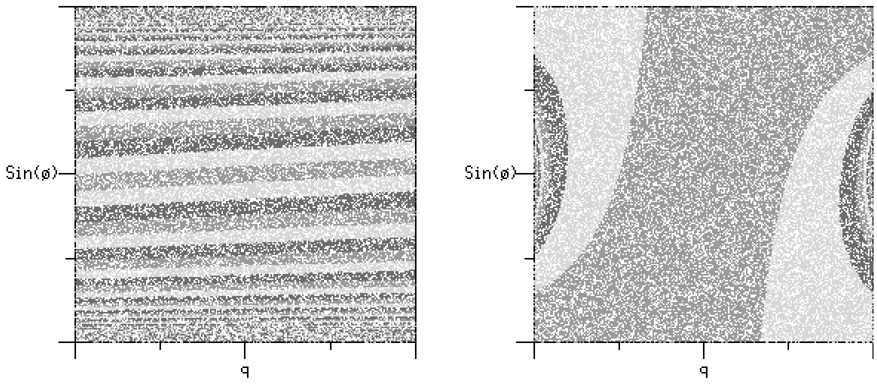

Figure 3: The Poincare section for a flight in the a=10 stadium.

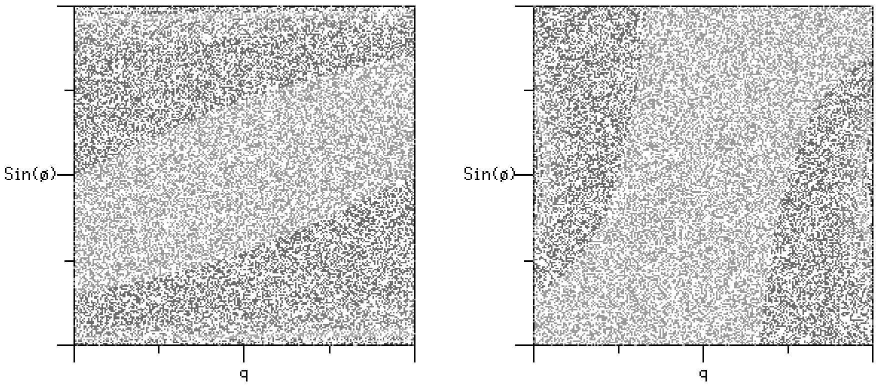

Figure 4: The Poincare section for a flight in the a=1 stadium.

5 Periodic orbits for the stadium billiard.

Several methods have been proposed for finding periodic orbits in the stadium billiard. The methods typically takes advantage of the used symbolic dynamics. [Bih92] uses a method where they places an atom on the right segment of the billiard for each symbol in the symbol-string for the wanted periodic orbit. Then they draw lines from atom to atom in the right order and tries to minimise the difference between the outgoing and incoming angle of these lines, formally written as a function $v = \sum_{i=1}^{n}(\theta_i^+ - \theta_i^-)^2$ which is to be minimised by use of forces $F_i = C(\theta_i^+ - \theta_i^-)$, where C is a constant, which sign depends on, whether the i-th point is in an arc or on a line. So their search has one parameter for each bounce in the stadium. [Cv95a] uses a method proposed by [Han95] which is a search in the symbol plan after the wanted orbit. I have chosen to use a refined version of the method from [Bih92], which takes advantage of the integrability of the two parts of the stadium. For each symbol in the symbolic dynamics from [Cv95a], there is at most a two parameter search, one for each endpoint in the

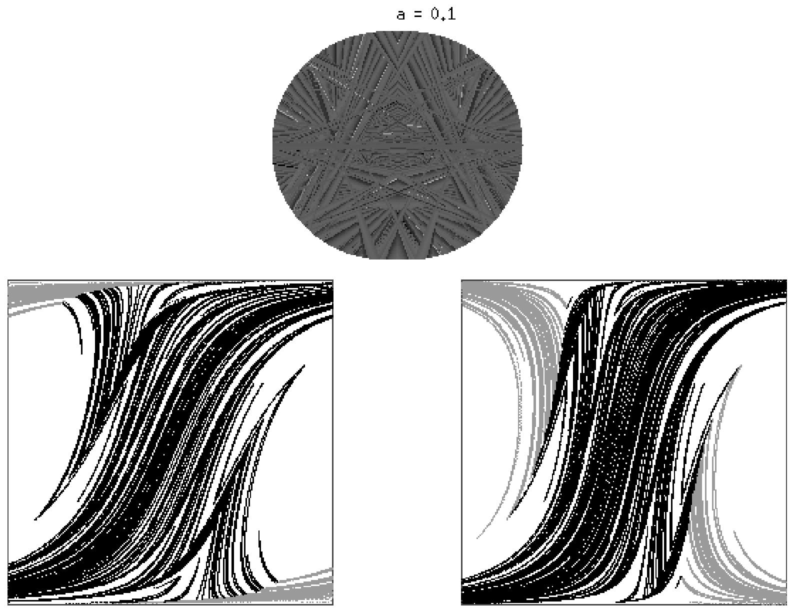

Figure 5: Some of the unstable manifolds of the $\overline{\rho_0\alpha_0}$ orbit. Left when leaving the Straight line section and right when leaving an arc. They are calculated at a=0.1

rotation/free flight. So in order to find the whispering gallery orbit i.e $\overline{\rho_{-a}\alpha_0}$ for $a \to \infty$ (fundamental domain symbolic dynamics), this method only uses a four parameter search, two parameters for each half circle, (since all searches are done in the closed, full stadium after a translation of the symbolic dynamics from unfolded fundamental domain to full closed). Since this method uses the symbolic dynamics from [Cv95a], which has no geometric pruning rules, the only pruning to be accounted for is dynamical pruning. And that is simply taken care of, since, if an orbit is pruned one atom eventually during the iteration leaves it's allowed position, i.e a point on one of the straight lines gets a position $|x_1| > a$ and a point on the circles enters nonexistent inner half circles.

6 Summation of the Periodic Orbits

When dealing with a large number of periodic orbits, it is to longish a task to try to perform the wanted calculations (i.e. Zeta-functions and/or Fredholm determinants) by hand. Since I have chosen to include orbits up to a number of 6 bounces on the fundamental domain closed stadium side walls, I have at most 378 different orbits to sum for each a, depending on the stadium parameter: the longer the stadium, the smaller the number of pruned orbits. See figure 6. The use for the periodic orbits was in this case to calculate the Lyapunov exponent.

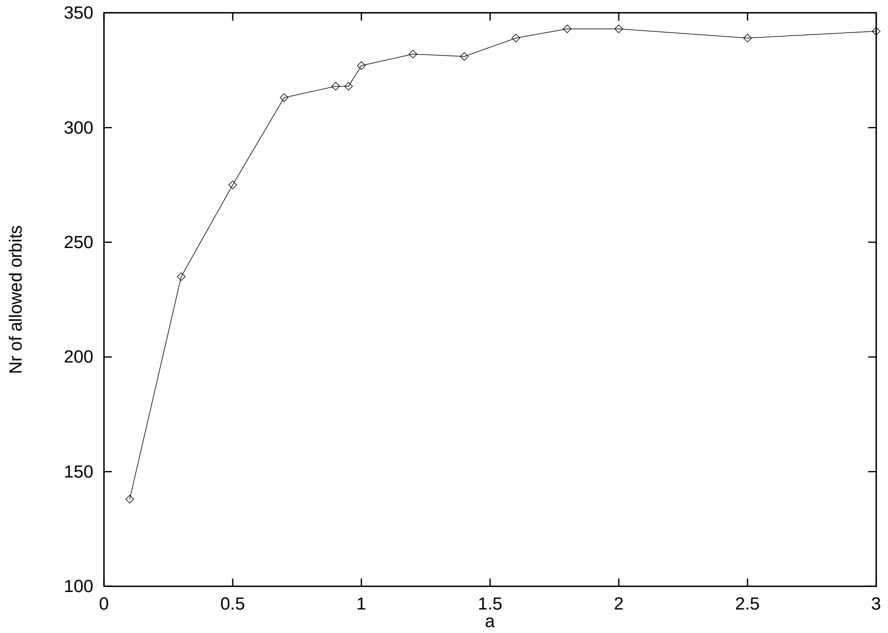

Figure 6: The number of admissible orbits up to period length 6 as a function of the stadium parameter.

The periodic theory equation for this in maps and in 2d-Hamiltonian flows is ([Cvi95],chapter 7).

$$\lambda = \frac{1}{\langle T \rangle_\zeta} \sum{}' (-1)^{k+1} \frac{\log |\Lambda_{p_1}| + \cdots + \log |\Lambda_{p_k}|}{|\Lambda_{p_1} \cdots \Lambda_{p_k}|} \tag{12}$$

which comes from the cycle averaging formula. To use this equation, we calculate the zeta function

$$1/\zeta = G = \prod (1 - t_p) \tag{13}$$

Where t_p is set to

$$t_p = \frac{e^{\beta \ln |\Lambda_p| - sT_p}}{|\Lambda_p|} \tag{14}$$

We have to numerically take care not to sum products of cycles longer than the used period 6. This is done by multiplicating recursively

$$\frac{1}{\zeta_{(i)}} = \frac{1}{\zeta_{(i-1)}} (1 - t_p z^{n_p}) \tag{15}$$

and truncating the expansion to

$$\frac{1}{\zeta^{(6)}} = 1 - \sum_{n=1}^{6} c_n z^n \tag{16}$$

and then setting z=1. We now use

$$\left.\frac{\partial G}{\partial \beta}\right|_{\beta=0,s=0} = \sum{}'(-1)^{k+1}\frac{\log|\Lambda_{p_1}| + \cdots + \log|\Lambda_{p_k}|}{|\Lambda_{p_1}\cdots\Lambda_{p_k}|} \tag{17}$$

and

$$\left.\frac{\partial G}{\partial s}\right|_{\beta=0,s=0} = \langle T\rangle_\zeta \tag{18}$$

The Lyapunov exponent is then given as

$$\lambda = -\frac{\left.\frac{\partial G}{\partial \beta}\right|_{\beta=0,s=0}}{\left.\frac{\partial G}{\partial s}\right|_{\beta=0,s=0}} \tag{19}$$

7 The Lyapunov Exponent Calculated from Cycles

Each of the two derivatives in (19) are easily calculated numerically, and the Lyapunov exponent is found. The cycle averaged Lyapunov exponent as a function of the stadium parameter is seen in figure 7 with the earlier shown graph which was calculated by following the Jacobian on a very long trajectory.

8 Conclusion

In this project I have calculated the Lyapunov exponent for the stadium billiard for a range of different values of a. I have done the calculations in two ways, by a standard brute force method, the evaluation of the Jacobian around the trajectory, and by a sophisticated, new method from the periodic orbit theory.

Even though I haven't used that many periodic orbits per value of a, the two curves for the Lyapunov exponent as a function of a follow each other nicely. See figure 7. Furthermore I have reproduced the Lyapunov exponent at a=1.6 from [Bih92]. I have made a program to calculate the washboard diffusion, but I haven't had the time to use it in a way to get usable results (i.e. FWHM values for Gaussians).

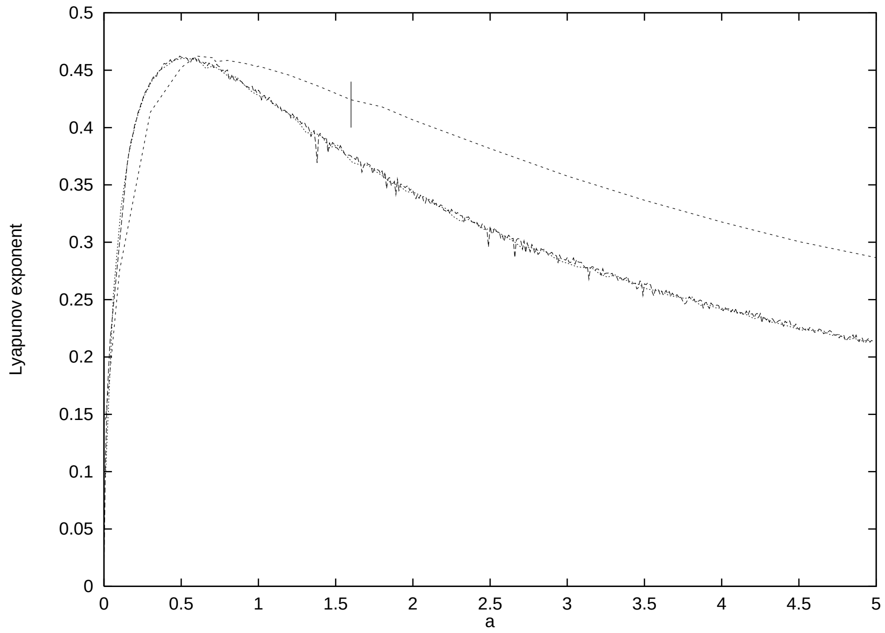

Figure 7: The Lyapunov exponent for the stadium as a function of a, calculated by the cycle averaging formula (dotted line). The maximal Lyapunov exponent is reached at a= 0.6 +/- 0.1. At a=1.6 is shown a vertical line. It is the value of the Lyapunov exponent from [Bih92], which is averaged over cycles up to length 13.

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A The Programs

A number of computer programs very written for this project.

1. A procedure with a stadium simulator, which given the position and momentum vectors just after a bounce, returns the position and momentum vectors right after the next bounce. This simulator was implemented using pure vector calculations i.e. it *eschewed* trigonometric functions to make it much faster.

2. A program that evaluated the Jacobian on a trajectory in the stadium using the simulator and returned the *Lyapunov Exponent* $\overline{\lambda}$ averaged on a given number of trajectories and a given number of bounces for each trajectory.

3. A program that calculated the x_2 positions of a given number of particles at given times and put the positions in bins of a given size, for later calculation of washboard diffusion. Unfortunately I never had time to do serious experiments with this program.

4. A program that finds all symbols strings up to period 6 in the fundamental domain, and leaves out nonprimecycles and permuted cycles.

5. A procedure, that given a symbol string in the fundamental domain, calculated the symbolstring in the full stadium.

6. A procedure, that given a full stadium symbol string finds the corresponding periodic orbit (and tests if the symbol string is pruned for the given a). Then returns the stability $|\Lambda_p|$ and Flight time T_p of the orbit.

7. A program that calculates the zeta function for orbits read from a file. Then calculates the Lyapunov exponent by differentiation of the zeta function by two variables.

8. A program that graphically display the Poincare section of the orbit in different colours according to number of bounces in arc/on lines. When this program is used with a large number ($\approx 10^4$) of particles with starting positions interpolated continuously between to positions very close (i.e. $\delta\mathbf{x} \approx 10^{-10}$) to the $\overline{00}$ orbit (the horizontal orbit from arc-end to arc-end), the picture obtained is the unstable manifolds of this orbit.