Deterministic Diffusion in the Zig-Zag Map

Termpaper by
Jakob Kisbye Dreyer (260373)

May 18, 1999

Introduction

The purpose of this termpaper is to investigate 1-dimensional deterministic diffusion in the zig-zag map. I calculate the diffusion constant for several specific values of the parameter Λ (– see below), with periodic orbit theory, and test these results in a numerical procedure.

1 The Map

The zig-zag map is for $x \in [0,1]$ is given by:

$$\hat{f}(x) = \begin{cases} \Lambda x & x \in [0, 1/4 + 1/(4\Lambda)] \\ -\Lambda x + \frac{\Lambda+1}{2} & x \in [1/4 + 1/(4\Lambda), 3/4 - 1/(4\Lambda)] \\ \Lambda x + 1 - \Lambda & x \in [3/4 - 1/(4\Lambda), 1] \end{cases}$$

$\Lambda > 1$ is required for diffusion to take place. The map is extended to the real axis by the rule:

$$\hat{f}(x + \hat{n}) = \hat{f}(x) + \hat{n} \quad \hat{n} \in \mathbf{Z}$$

The corresponding circle map is defined on the unit circle by $f(x) = \hat{f}(x)(\text{mod } 1)$. To every trajectory in $\hat{f}(x)$ there is a corresponding trajectory in $f(x)$.
The diffusion constant,D can be defined by the Einstein relation

$$\langle (\hat{x}_n - \langle \hat{x}_n \rangle)^2 \rangle = 2Dn,$$

and can be expressed in terms of derivatives of the leading eigenvalue of the zeta function. The zetafunction is given as

$$\frac{1}{\zeta(z,\beta)} = \prod_p (1 - t_p)$$

– the product extends over all periodic orbits (cycles),p, in the map and t_p is the cycle weight of the cycle, p. This infinite product can then be expanded as a polymonial:

$$\frac{1}{\zeta(z,\beta)} = 1 - t_{p_1} - \cdots - (t_{p_1 p_2} - t_{p_1} t_{p_2}) - \cdots$$

with $t_p = \frac{z_p^n e^{\beta \hat{n}_p}}{\Lambda_p}$. Since the map is piecewise linear, we can use the formula

$$t_{p_1 p_2} = t_{p_1} t_{p_2} \ .$$

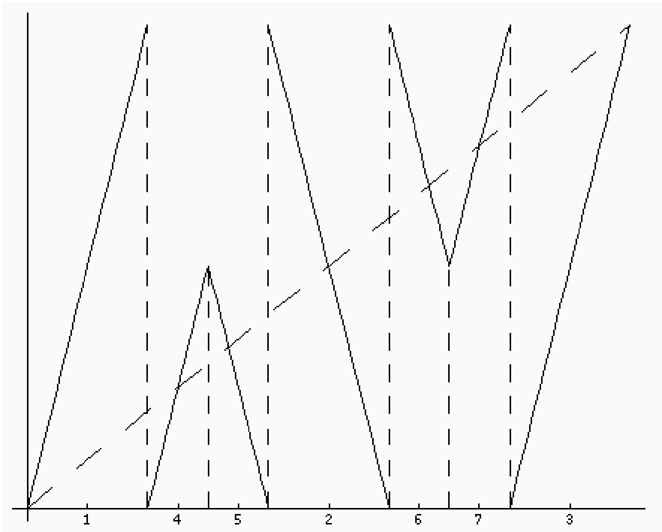

Figure 1: The map for the parameter $\Lambda = 5$. The symbolic dynamics are defined by the subintervals on the x-axis. For this special value of Λ the partition is actually different, as will be discussed later in this paper.

We have in this case a flowconservation equation of the form

$$\frac{1}{\zeta(1,0)} = 0 \ .$$

From which we get

$$\langle n \rangle = -z \frac{\partial}{\partial z} \frac{1}{\zeta(0,z)} \bigg|_{z=1} , \tag{1}$$

$$\langle \hat{n}^2 \rangle = -\frac{\partial^2}{\partial \beta^2} \frac{1}{\zeta(\beta,1)} \bigg|_{\beta=0} , \tag{2}$$

$$D = \frac{\langle \hat{n}^2 \rangle}{2\langle n \rangle} \ . \tag{3}$$

2 Diffusion Constants

In order to calculate the diffusion constants in the Zig Zag map, we must partition the phasespace (the unit interval) into a markov partition. The image of any partition must be *onto* a union of partions. This is accomplished by letting the *critical point* i.e. the highest point on the map be mapped on a interval border.

$$a = f(\frac{\Lambda + 1}{4\Lambda}) = \frac{\Lambda + 1}{4}$$

– or equivalently

$$\Lambda = 4a - 1 \ .$$

2.1 a integer.

The first case is a being an integer. The unit interval is divided into $4a - 1$ sections, $\mathcal{M}_k$. In this case the symbolic dynamics are unresticted, since the image of any section is the intire unit interval. The topological zeta function is therefore given as

$$\frac{1}{\zeta} = 1 - t_1 - t_2 - \ldots - t_{4a-1}$$

With the cycle weigths being:

$$\begin{aligned}
&t_1 = t_2 = t_3 = \tfrac{z}{\Lambda} \\
&t_4 = t_5 = \tfrac{z}{\Lambda} e^{\beta} \\
&t_6 = t_7 = \tfrac{z}{\Lambda} e^{-\beta} \\
&\quad\vdots \\
&t_{4k} = t_{4k+1} = \tfrac{z}{\Lambda} e^{k\beta} \qquad k = 1 \ldots a-1 \\
&t_{4k+2} = t_{4k+3} = \tfrac{z}{\Lambda} e^{-k\beta} \\
&\quad\vdots
\end{aligned}$$

The cycles are naturally grouped together in bundels of 4:

$$t_{4k} + t_{4k+1} + t_{4k+2} + t_{4k+3} = 4\frac{z}{\Lambda}\cosh k\beta \ .$$

Finally the zetafunction can be rewritten as

$$\frac{1}{\zeta(\beta, z)} = 1 - 3\frac{z}{\Lambda} - 4\frac{z}{\Lambda}\sum_{k=1}^{a-1}\cosh k\beta \ , \tag{4}$$

from which we obtain, with the use of eqns. 2, 1 and eqn. 3

$$\langle n \rangle = -\frac{3 - 4(a-1)}{\Lambda} = 1 \ ,$$

and

$$\langle \hat{n}^2 \rangle = \frac{4}{\Lambda}\sum_{k=1}^{a-1} k^2 = \frac{2a(a-1)(2a-1)}{3\Lambda} \ .$$

This gives the diffusion constant

$$D = \frac{\langle \hat{n}^2 \rangle}{2\langle n \rangle} = \frac{a(a-1)(2a-1)}{3\Lambda} = \frac{(\Lambda^2 - 1)(\Lambda - 3)}{96\Lambda}$$

– that hold for $\Lambda = 4a - 1,\ a$ integer.

2.2 Other values of Λ

In the following a is considered to be in the interval $[1, 2]$. (The Cases $a = 1, 2$ are dealt with previously.) The condition that a is mapped onto an interval border gives us that a must satisfy relationships like $\hat{f}(\frac{\Lambda+1}{4\Lambda}) - 1 = f(\frac{\Lambda+1}{4\Lambda}) = \frac{1}{\Lambda}$, where $\frac{1}{\Lambda}$ is the interval border between sections $\mathcal{M}_1, \mathcal{M}_4$. This gives:

$$\frac{\Lambda(\Lambda+1)}{4} = 1 + \Lambda \Rightarrow \Lambda^2 - 3\Lambda - 4 = 0$$

Which has the expanding solution $\Lambda = 4$. This case is dealt with in the Chaos Webbook, and is not included here.

2.2.1

The Next case considered is determined by the markovproperty

$$f(\mathcal{M}_4) = f(\mathcal{M}_5) = \mathcal{M}_1 \cup \mathcal{M}_4$$

and

$$f(\mathcal{M}_6) = f(\mathcal{M}_7) = \mathcal{M}_3 \cup \mathcal{M}_7.$$

(The image of $\mathcal{M}_6$ and $\mathcal{M}_7$ follow by the symmetry of the map)
The value of a or equivalently Λ is determined by the following relation:

$$f(1/4 + 1/(4\Lambda)) = 1/4 + 1/(4\Lambda) - 1 \Rightarrow \Lambda^2 - 4\Lambda - 1 = 0$$

The equation has the expanding solution

$$\Lambda = 2 + \sqrt{5} \ \Rightarrow \ a = 3/4 + \sqrt{5}/4$$

The symbolic dynamics in this case are such that the following subsequences are forbidden:

43,42,45,46,47;53,52,55,56,57;61,62,66,64,65;71,72,76,74,75.

This can be incorpotated in an alphabet. Such an aphabet could be

$$\mathcal{A} = \{1, 2, 3, 41, 441, \ldots, 51, 541, 5441, \ldots, 63, 673, 6773, \ldots, 73, 773, \ldots, \overline{4}, \overline{7}\}.$$

Note that this alphabet is infinite and the periodic points, $\overline{4}$ and $\overline{7}$, are explicitly included in the list since the grammar does not exclude these. The zetafunction is in this case given as

$$\frac{1}{\zeta} = (1 - t_2 - \sum_{k=0} t_{4^k 1} - \sum_{k=0} t_{54^k 1} - \sum_{k=0} t_{67^k 3} - \sum_{k=0} t_{7^k 3})(1 - t_4)(1 - t_7) \qquad k = 0, 1, 2, \ldots$$

We can simplify this by noting that $\frac{t_{4^{k+1}1}}{t_{4^k 1}} = t_4$, since $t_{p_1 p_2} = t_{p_1} t_{p_2}$. This means the sum can be represented as geometric series, with $\sum t_{4^k 1} = \frac{t_1}{1-t_4}$. Ofcourse the same argument works with all the above sums. The zetafunction is now

$$\frac{1}{\zeta} = (1 - t_2 - \frac{t_1}{1 - t_4} - \frac{t_{51}}{1 - t_4} - \frac{t_3}{1 - t_7} - \frac{t_{63}}{1 - t_7})\,(1 - t_4)\,(1 - t_7)$$

which simplifies to

$$\begin{aligned} \frac{1}{\zeta} &= 1 - t_1 - t_2 - t_3 - t_4 - t_7 - t_{51} - t_{63} \\ &\quad + t_3\, t_4 + t_{63}\, t_4 + t_{51}\, t_7 + t_1\, t_7 + t_2\, t_7 + t_2\, t_4 + t_4\, t_7 - t_2\, t_4\, t_7 \end{aligned}$$

The cycle weights are

$$\begin{aligned} t_1 = t_2 = t_3 &= \frac{z}{\Lambda} \\ t_4 = t_5 &= \frac{z}{\Lambda}\, e^{\beta} \\ t_7 = t_6 &= \frac{z}{\Lambda}\, e^{-\beta} \\ t_{51} &= \frac{z^2}{\Lambda^2}\, e^{\beta} \\ t_{63} &= \frac{z^2}{\Lambda^2}\, e^{-\beta} \end{aligned}$$

This gives

$$\frac{1}{\zeta(z,\beta)} = 1 - \frac{z}{\Lambda}(3 + 2\cosh\beta) + \frac{z^2}{\Lambda^2}(1 + 2\cosh\beta) + \frac{z^3}{\Lambda^3}$$

The diffusion constant is calculated with the formulas 1, 2 and 3, giving

$$\begin{aligned}
\langle n \rangle &= \frac{5}{\Lambda} - \frac{6}{\Lambda^2} - \frac{3}{\Lambda^3}\,, \\
\langle \hat{n}^2 \rangle &= \frac{2}{\Lambda} - \frac{2}{\Lambda^2}\,, \\
D &= \frac{\Lambda^2 - \Lambda}{5\Lambda^2 - 6\Lambda - 3} = \frac{1}{2\sqrt{5}}\,.
\end{aligned}$$

2.2.2

The dynamics in the next case are determined by the markov property

$$f(\mathcal{M}_4) = f(\mathcal{M}_5) = \mathcal{M}_1 \cup \mathcal{M}_4 \cup \mathcal{M}_5$$

– that is: a is mapped onto the intervalborder between $\mathcal{M}_5$ and $\mathcal{M}_2$. Letting x denote the intervalborder between $\mathcal{M}_5$ and $\mathcal{M}_2$, x can be found from the linear relation

$$-\Lambda x + \frac{\Lambda + 1}{2} = 1 \;\Rightarrow\; x = \frac{\Lambda - 1}{2\Lambda}$$

The relation to determine Λ is now

$$f(a) = \frac{\Lambda}{4} + \frac{1}{4} - 1 = \frac{\Lambda - 1}{2\Lambda}$$

which has the solution

$$\Lambda = \frac{5 + \sqrt{17}}{2} \qquad \Rightarrow \qquad a = \frac{7 + \sqrt{17}}{8}$$

The symbolic dynamics in this case are such that the following subsequences are forbidden:

43,42,46,47; 53,52,56,57; 61,62,64,65; 71,72,74,75;

We will however still try to find an alphabet that describes the symbolic dynamics. Such an alphabet could be the following:

$$\begin{aligned}
\mathcal{A} = \{&1, 2, 3, 41, 51, 441, 451, 551, 541, \ldots, 63, 73, 673, \ldots, \\
&\overline{4}, \overline{5}, \overline{45}, \overline{445}, \ldots, \overline{6}, \overline{7}, \overline{67}, \overline{667}, \ldots\}
\end{aligned} \tag{5}$$

Where the dots in the first line represent all combinations of 4's and 5's (or 6's and 7's) followed by 1 (or 3), and the dots in the last line represent all prime combinations of 4 and 5 (or 6 and 7). The second line contains explicitly the periodic points of the map which are not excluded by the grammar.
At this point we introduce the following notation:

$$[x, y]^k = \textit{All combinations of } x \textit{ and } y \textit{ of length } k.$$ [1]

In this notation the alphabet becomes

$$\mathcal{A} = \{2, [4,5]^k 1, [6,7]^k 3, \overline{4}, \overline{5}, \overline{45}, \overline{445}, \ldots, \overline{6}, \overline{7}, \overline{67}, \ldots\} \quad k = 0, 1, 2, \ldots$$

[1] I don't know if there exist any standart terminology.

Note that this new notation does not apply on the periodic orbits since these only contain prime combinations of 4, 5, 6 and 7. The corresponding zetafunction can be expressed as a product of the orbits from the first line in 5) and the orbits from the second line. Letting Φ denote the orbits defined in the first line and Θ denote the periodic orbits in the second line, the zeta function can be written as

$$\frac{1}{\zeta} = \prod_{p\in\Phi}(1-t_p)\prod_{q\in\Theta}(1-t_q)$$

At this point we note that Θ devides into two disjoint subsets in which the grammar is unrestricted, namely periodic orbits that consists of combinations of 4 and 5, and combinations of 6 and 7, respectively. This gives

$$\begin{aligned}
\frac{1}{\zeta} &= \prod_{p\in\Phi}(1-t_p)(1-t_4-t_5)(1-t_6-t_7) \\
&= (1-t_2-t_1\sum_{k=0}^{\infty}\sum_{p\in[45]^k} t_p - t_3\sum_{k=0}^{\infty}\sum_{p\in[67]^k} t_p)\,(1-t_4-t_5)(1-t_6-t_7).
\end{aligned}$$

By expanding the product in Φ level by level in the alphabet, and using the property $t_{p_1p_2} = t_{p_1}t_{p_2}$ we find that

$$\begin{aligned}
\sum_{p\in[xy]^k} t_p &= \sum_{p\in[yx]^{k-1}} t_{xp}+t_{yp} \\
&= (t_x+t_y)\sum_{p\in[yx]^{k-1}} t_p \\
&= (t_x+t_y)^k
\end{aligned}$$

–the last equality follow recursively. This enables us to rewrite the zetafunction using geometric series

$$\begin{aligned}
\frac{1}{\zeta} &= (1-t_2-\frac{t_1}{1-t_4-t_5}-\frac{t_3}{1-t_6-t_7})\,(1-t_4-t_5)(1-t_6-t_7) \\
&= 1-t_1-t_2-t_3-t_4-t_5-t_6-t_7+t_1t_6+t_1t_7 \\
&\quad + t_4t_3+t_5t_3+t_2t_4+t_2t_5+t_2t_6+t_2t_7+t_4t_6+t_4t_7 \\
&\quad + t_5t_6+t_5t_7-t_2t_4t_6-t_2t_4t_7-t_2t_5t_6-t_2t_5t_7
\end{aligned}$$

using geometric series. The cycle weights are computed as usual and inserted to give

$$\frac{1}{\zeta(z,\beta)} = 1-\frac{z}{\Lambda}(3+4\cosh\beta)+\frac{z^2}{\Lambda^2}(4+8\cosh\beta)-4\frac{z^3}{\Lambda^3}\ . \tag{6}$$

From eqn. 6 we get the following expressions, according to eqns. 1, 2 and 3:

$$\begin{aligned}
\langle\hat{n}^2\rangle &= 4/\Lambda-8/\Lambda^2\ , \\
\langle n\rangle &= 7/\Lambda-24/\Lambda^2+12/\Lambda^3\ , \\
D &= \frac{2\Lambda^2-4\Lambda}{7\Lambda^2-24\Lambda+12} = \frac{2}{17\sqrt{17}}\ .
\end{aligned}$$

2.2.3

The dynamics in the next case is determined from the markovproperty $f(\mathcal{M}_4) = \mathcal{M}_1 \cup \mathcal{M}_4 \cup \mathcal{M}_5 \cup \mathcal{M}_2$. The value of Λ where this happens can be tetermined from the relation

$$\Lambda/4 - 3/4 = \frac{\Lambda + 1}{2\Lambda} \quad \Rightarrow \quad \Lambda = \frac{1}{2}(5 + \sqrt{33})$$

– as usual we've picked only the expanding solution to the quadratic equation.
The corresponding alphabet becomes (in the notation from the previous paragraph)

$$\mathcal{A} = \{[4,5]^k 1, [4,5]^k 2, [6,7]^{k+1} 2, [6,7]^k 3; \overline{4}, \overline{5}, \overline{45}, \ldots, \overline{6}, \overline{7}, \overline{67}, \ldots\}$$

for $k = 0, 1, 2, 3, \ldots$ Bearing in mind the results from the last paragraph we can write down the zetafunction:

$$\begin{aligned} \frac{1}{\zeta} &= (1 - \frac{t_1}{1 - t_4 - t_5} - \frac{t_2}{1 - t_4 - t_5} - \frac{t_2}{1 - t_6 - t_7} + t_2 - \frac{t_2}{1 - t_6 - t_7}) \\ &\quad \times (1 - t_4 - t_5)(1 - t_6 - t_7). \end{aligned}$$

Note that we have added and subtracted an extra t_2. Rewriting the zetafunction to

$$\begin{aligned} \frac{1}{\zeta} &= 1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_6 - t_7 \\ &\quad + t_1(t_6 + t_7) + t_3(t_4 + t_5) + t_4(t_6 + t_7) + t_5(t_6 + t_7) \\ &\quad + t_2(t_4 + t_5)(t_6 + t_7) \end{aligned}$$

we arrive at

$$\frac{1}{\zeta(z, \beta)} = 1 - \frac{z}{\Lambda}(3 + 4\cosh\beta) + \frac{z^2}{\Lambda^2}(4 + 4\cosh\beta) + 4\frac{z^3}{\Lambda^3} \ ,$$

and calculate the diffusion constant the usual way...

$$\begin{aligned} \langle n \rangle &= 7/\Lambda - 16/\Lambda^2 - 12/\Lambda^3 \\ \langle \hat{n}^2 \rangle &= 4/\Lambda - 4/\Lambda^2 \\ D &= \frac{2\Lambda^2 - 2\Lambda}{7\Lambda^2 - 16\Lambda - 12} \\ &= 1/8 + 5/88\sqrt{33} \end{aligned}$$

2.2.4

The next case is determined from the markov property

$$f(\mathcal{M}_4) = \mathcal{M}_1 \cup \mathcal{M}_4 \cup \mathcal{M}_5 \cup \mathcal{M}_2 \cup \mathcal{M}_6$$

This determines a value Λ by the relation:

$$\hat{f}(1/4 + 1/(4\Lambda)) - 1 = 3/4 - 1/(4\Lambda) \Rightarrow \Lambda^2 - 6\Lambda + 1 = 0$$

The expanding solution is $\Lambda = 3 + 2\sqrt{2}$.

We will now leave the path we've been proceeding so far, and pursue another path. We will bring forth the Markovgraph in fig. 2. The topological zetafunction will then be given as the sum of all closed nonintersecting loops on this graph, with each loop carrying a minus sign. Before we disect the markovgraph we note that the orbits connect 3 nodes. Any nonselfintersecting walk on the graf in maximally of length 3. In fig. 2 we have (for simplicity we let $t_p = p$):

The closed walks...

Loops of length 1: $\{1,2,3,4,5,6,7\}$.

Loops of length 2: $\{41,42,51,52,62,63,72,73,65\} = \{(4,5)(1,2),(6,7)(2,3),65\}$. $(p_1,p_2)Q$ is a shorthand notation for the combinations p_1Q, p_2Q and Q is any set of combinations.

Loops of length 3: $\{(4,5)6(2,3),(6,7)5(1,2)\}$

Products of nonintersecting loops...

The products of two distinct loops: $\{(1,2,3)\times(4,5),(1,2,3)\times(6,7),(4,5)\times(6,7)\,,\,(1,2,3)\times 65,(4,5)\times(6,7)(2,3),(6,7)\times(4,5)(1,2)\}$. These terms will carry a plus-sign in the zeta function.

Products of three loops: $\{(1,2,3)\times(4,5)\times(6,7)\}$. These terms all carry minus-sign.

We find that the most of these loops appear in shadowing combinations like $[t_{41} - t_4t_1]$, and in that manner most terms cancel the surviving zetafunction is then

$$\begin{aligned}\frac{1}{\zeta} &= 1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_6 - t_7 \\ &\quad + t_3(t_4+t_5) + t_1(t_6+t_7) + t_4(t_6+t_7) + t_5t_7 \\ &\quad + t_4t_{72} - t_1t_5t_7 - t_3t_4t_6 \\ &= 1 - \frac{z}{\Lambda}(3+4\cosh\beta) + \frac{z^2}{\Lambda^2}(3+4\cosh\beta) - \frac{z^3}{\Lambda^3}\end{aligned}$$

The diffusion constant is then given by the equations

$$\begin{aligned}\langle \hat{n}^2\rangle &= 4/\Lambda - 4/\Lambda^2\ , \\ \langle n\rangle &= 3/\Lambda^3 - 14/\Lambda^2 + 7/\Lambda\ , \\ D &= \frac{2\Lambda^2 - 2\Lambda}{7\Lambda^2 - 14\Lambda + 3} = \frac{1}{4}\sqrt{2}\end{aligned}$$

2.2.5

The dynamics are determined by the markovproperty

$$f(\mathcal{M}_4) = f(\mathcal{M}_5) = \mathcal{M}_1 \cup \mathcal{M}_4 \cup \mathcal{M}_5 \cup \mathcal{M}_2 \cup \mathcal{M}_6 \cup \mathcal{M}_7$$

This determines Λ to be $\frac{7+\sqrt{33}}{2}$. The markovgraph for this system is not included here since it resembles fig. 2. The only difference is in the transistions between the two lower nodes on the graph. On fig. 2 the arrow '5' and the arrow '6' is replaced with double arrow '4 5' and double arrow '6 7'.

By counting the loops in the same way as descriped above we get the following zeta function:

$$\begin{aligned}\frac{1}{\zeta} &= 1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_6 - t_7 + t_3(t_4+t_5) + t_1(t_6+t_7) \\ &= 1 - \frac{z}{\lambda}(3+4\cosh\beta) + 4\frac{z^2}{\Lambda^2}\cosh\beta\ .\end{aligned}$$

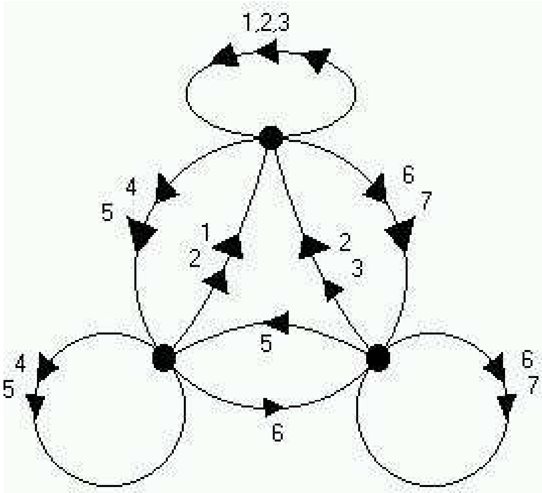

Figure 2: A: Markovgraph in the case $\Lambda = 3 + 2\sqrt{2}$.

With the Diffusionconstant being

$$
\begin{aligned}
\langle \hat{n}^2 \rangle &= 4/\Lambda - 4/\Lambda^2 \\
\langle n \rangle &= 7/\Lambda - 8/\Lambda^2 \\
D &= \frac{2\Lambda - 2}{7\Lambda - 8} = \frac{33 + \sqrt{33}}{132}
\end{aligned}
$$

2.3 Special Case: $\Lambda = 5$.

To find the diffusionconstant in this case we need to repartition the unit interval.

$$\mathcal{M}_2 \longrightarrow \mathcal{M}_8 \cup \{1/2\} \cup \mathcal{M}_9$$

Where $\{1/2\}$ contains only the fixed point $x = 1/2$, and is denoted $\mathcal{M}_2$. $\mathcal{M}_2$ is mapped only onto it self, and all trajectories that end in this interval will stay here forever. The other partitions are not altered. The markovproperties are:

$$f(\mathcal{M}_4) = \mathcal{M}_1 \cup \mathcal{M}_4 \cup \mathcal{M}_5 \cup \mathcal{M}_8 \cup \mathcal{M}_2 \ ,$$

$$f(\mathcal{M}_5) = \mathcal{M}_1 \cup \mathcal{M}_4 \cup \mathcal{M}_5 \cup \mathcal{M}_8$$

and

$$f(\mathcal{M}_8) = \mathcal{M}_3 \cup \mathcal{M}_6 \cup \mathcal{M}_7 \cup \mathcal{M}_9$$

with similar properties for the partitions $\mathcal{M}_6, \mathcal{M}_7$ and $\mathcal{M}_9$.

We here face the problem that eventhough transitions between the subintervals $\mathcal{M}_8$ and $\mathcal{M}_9$ are allowed there are no fixed points like t_{89} *ect.* This will give rise to a pole in the zetafunction. We will not try to wrtie down an alphbet in which the symbolic dynamics are unrestricted, but here is an out line of what happens:
We split the symbolseqeunces that start with 4 or 5 into three groups.

the first go imeadiately to 1

The second contains the same number of 8's and 9's. This group can contain a hole subtree of 6,7 of arbitrary length, but terminates with 1

The third group have one more 8's than 9's and must therefore end must end with 3.

No sequences that start with 4 or 5 and terminate with 1 or 3 can have more 9's than 8's. A similar picture arises with sequenses that start in 6,7, and we have 6 different group of cycles. We have:

$$\frac{1}{\zeta} = \left[1 - \sum_{1st} t_p - \sum_{2nd} t_q - \ldots\right] \times \prod t_{\overline{p}}$$

–where the sums go over orbits in each group. From different groups we get:

$$\begin{aligned}
\sum_{1st} t_p &= \frac{t_1}{1 - t_4 - t_5} \\
\sum_{2nd} t_p &= t_1 \sum_{k=0} [45]^k \sum_{l=1} 8^l 9^l \sum_{j=0} [6,7]^j = \frac{t_1 t_8 t_9}{(1 - t_4 - t_5)(1 - t_6 - t_7)(1 - t_8 t_9)} \\
\sum_{3rd} t_p &= t_1 \sum_{k=0} [45]^k \sum_{l=1} 8^l 9^{l-1} \sum_{j=0} [6,7]^j = \frac{t_1 t_8}{(1 - t_4 - t_5)(1 - t_6 - t_7)(1 - t_8 t_9)}
\end{aligned}$$

The product over the periodic orbits $\overline{2}, \overline{4}, \overline{5}, \overline{6}, \overline{7}, \ldots$ becomes as we have seen previously $(1 - t_2)(1 - t_4 - t_5)(1 - t_6 - t_7)$. We get the zetafuntion

$$\begin{aligned}
\frac{1}{\zeta} &= \Bigg[1 - \frac{t_1}{1 - t_4 - t_5} - \frac{t_3}{1 - t_6 - t_7} - \frac{(t_1 + t_3) t_8 t_9}{(1 - t_4 - t_5)(1 - t_6 - t_7)(1 - t_8 t_9)} \\
&\quad - \frac{t_1 t_8}{(1 - t_4 - t_5)(1 - t_6 - t_7)(1 - t_8 t_9)} - \frac{t_3 t_9}{(1 - t_4 - t_5)(1 - t_6 - t_7)(1 - t_8 t_9)}\Bigg] \\
&\quad \times (1 - t_2)(1 - t_4 - t_5)(1 - t_6 - t_7) \\
&= \Big[1 - t_1 - t_3 - t_4 - t_5 - t_6 - t_7 + t_1(t_6 + t_7) + t_3(t_4 + t_5) \\
&\quad - \frac{(t_1 + t_3) t_8 t_9}{1 - t_8 t_9} - \frac{t_1 t_8}{1 - t_8 t_9} - \frac{t_3 t_9}{1 - t_8 t_9}\Big](1 - t_2) \;.
\end{aligned}$$

We set $t_8 = t_9 = t_2$, and get

$$\begin{aligned}
\frac{1}{\zeta} &= \Big[1 - t_1 - t_3 - t_4 - t_5 - t_6 - t_7 + t_1(t_6 + t_7) + t_3(t_4 + t_5)\Big](1 - t_2) \\
&\quad - \frac{(t_1 + t_3)(t_2^2 + t_2)}{1 + t_2}
\end{aligned}$$

This is however not correct as can be seen when evaluated (1,0) for $\Lambda = 5$. We find the same pole here as in sec. 3. See below. This is as good as it gets...

3 Symmetri Factorisation

The Zig-Zag–map is reflection symmetric and we can exploit this feature to factorize the zeta-function and work in the fundamental cell (fig. 3), instead of the full map. Any cycle in the full map, is either symmmetric or antisymmetric with respect to the operation of reflecting the unit interval through its center $x = 1/2$. By this we mean that the points on a cycle are mapped,

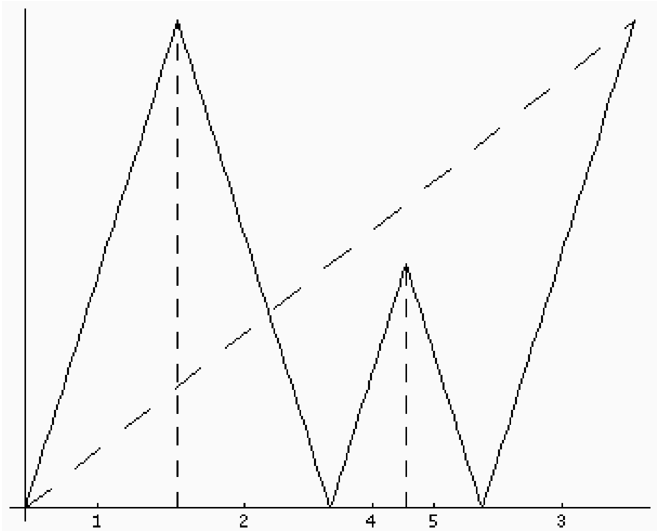

Figure 3: The map in the fundamental domain, for the value $\Lambda = 4$.

by the reflection, onto other points on the same cycle (symmetric) or points on a different cycle with the same length (antisymmetric). In the fundamental cell the cycles are called symmetric if the coresponding cycle in the full map is symmetric and antisymmetric if the coresponding cycle in the full map is anti symmetric.

When working in the fundamental cell, we find that any antisymmetric cycle coresponds to two cycles with the same weight in the full map if the parameter $\beta = 0$. Any symmetric orbit in the fundamental cell coresponds to a cycle with the double length. The only exception to this rule is the fixed point t_2 in the full map which is just mapped onto its self by the reflection operation. This fixed point is called the boundary cycle in the fundamental cell. The contribtions to the zeta function from each of these 3 categories of cycles is then given as follows:

Orbit:	Full Map		A_1 A_2
Symmtric:	$(1-t_p^2)$	$=$	$(1-t_p)(1+t_p)$
Antisymmetric:	$(1-t_p)^2$	$=$	$(1-t_p)(1-t_p)$
Boundary:	$(1-t_p)$	$=$	$(1-t_p)(1-0)$

We find the zetafunction in the full map to be a product of two zetafunctions $\frac{1}{\zeta_{A_1}}, \frac{1}{\zeta_{A_2}}$. The zetafunction in the fundamental cell is just $\frac{1}{\zeta_{A_1}}$.

The interval $[0, 1/2]$ is diveded into 5 subintervals and the partitions are labelled $\tilde{1}, \tilde{2}, \tilde{4}, \tilde{5}, \tilde{3}$, succesively. The boundarycycle is $t_{\tilde{3}}$. We will leave out the tilde's when no confusion can arise. The partitions $\tilde{2}, \tilde{3}$ act as reflectors, in the sense that if any trajectory hit these intervals, the coresponding trajectory in the full map is crossing the middle of the interval $[0, 1]$. It is aperent that any periodic orbit that has a uneven number of hits in either $\tilde{2}$ or $\tilde{3}$ is symmetric. In A_2 any shadowing combination will still work regardles of the change of sign in the symmetric cycles – except possibly those affected by the boundary cycle.

3.1 Let's Get Funky!

We let $\tilde{f}$ denote the map f in the fundamental cell. The first case ($\Lambda = 2 + \sqrt{5}$):
In the full map the property

$$f(\mathcal{M}_4) = f(\mathcal{M}_5) = \mathcal{M}_1 \cup \mathcal{M}_4 \ ,$$

coresponds to

$$\tilde{f}(\mathcal{M}_{\tilde{4}}) = \tilde{f}(\mathcal{M}_{\tilde{5}}) = \mathcal{M}_{\tilde{1}} \cup \mathcal{M}_{\tilde{2}} \cup \mathcal{M}_{\tilde{4}}$$

in the fundamntal cell. From this we can write down a complete alphabet in the fundamental cell

$$\tilde{\mathcal{A}} = \{3, 4^k 1, 4^k 2, 54^k 1, 54^k 2, \overline{4}\} \qquad k = 0, 1, 2, 3, \ldots$$

giving:

$$\begin{aligned}
\frac{1}{\zeta_{A_1}} &= \left(1 - t_3 - \frac{t_1 + t_2 + t_{51} + t_{52}}{1 - t_4}\right)(1 - t_4) \\
&= 1 - t_1 - t_2 - t_3 - t_4 - t_{51} - t_{52} + t_3 t_4 \\
&= 1 - 4/\Lambda - 1/\Lambda^2
\end{aligned}$$

– we imeadiately evaluate the zetafunctions in $z = 1$ and $\beta = 0$.
To find $\frac{1}{\zeta_{A_2}}$ we just multiply it by $(1 + t3)/(1 + t_3)$ to make the shadowing work in the same way as in $\frac{1}{\zeta_{A_1}}$. We get the result:

$$\begin{aligned}
\frac{1}{\zeta_{A_2}} &= \frac{1 - t_1 + t_2 + t_3 - t_4 - t_{51} + t_{52} - t_3 t_4}{1 + t_3} \\
&= \frac{1 - 1/\Lambda^2}{1 + 1/\Lambda} = 1 - 1/\Lambda \ ,
\end{aligned}$$

and we therefore get

$$1/\zeta(1, 0) = (1 - 1/\Lambda)(1 - 4/\Lambda - 1/\Lambda^2) \ ,$$

in agreement with the previous result.

In the next case we find the property in the fundamental cell ($\Lambda = (5 + \sqrt{17})/2$):

$$\tilde{f}(\mathcal{M}_{\tilde{4}}) = \tilde{f}(\mathcal{M}_{\tilde{5}}) = \mathcal{M}_{\tilde{1}} \cup \mathcal{M}_{\tilde{2}} \cup \mathcal{M}_{\tilde{4}} \cup \mathcal{M}_{\tilde{5}} \ .$$

This gives the alphabet:

$$\tilde{\mathcal{A}} = \{3, [4, 5]^k 1, [4, 5]^k 2, \overline{4}, \overline{5}, \overline{45}, \ldots\} \qquad k = 0, 1, 2, 3, \ldots \quad ,$$

and the zetafunction:

$$\begin{aligned}
\frac{1}{\zeta_{A_1}} &= \left(1 - t_3 - \frac{t_1 + t_2}{1 - t_4 - t_5}\right)(1 - t_4 - t_5) \\
&= 1 - t_1 - t_2 - t_3 - t_4 - t_5 + t_3 t_4 + t_3 t_5 \\
&= 1 - 5/\Lambda + 2/\Lambda^2 \ .
\end{aligned}$$

We find the other zeta function to be

$$\begin{aligned}
\frac{1}{\zeta_{A_2}} &= \frac{1 - t_1 + t_2 + t_3 - t_4 - t_5 - t_3 t_4 - t_3 t_5}{1 + t_3} \\
&= \frac{1 - 1/\Lambda - 2/\Lambda^2}{1 + 1/\Lambda} = 1 - 2/\Lambda \ .
\end{aligned}$$

–and we get

$$\frac{1}{\zeta} = (1-2/\Lambda)(1-5/\Lambda+2/\Lambda^2)\ .$$

The value of $\Lambda = (5+\sqrt{33})/2$ is greater than 1/2. This means that we need to repartition the interval $[0,1/2]$. The subintervals 4,5 are divided into 4,6,7,5, where the subintervals 6,7 now act as reflectors like 2,3. $\tilde{f}(\mathcal{M}_4) = [0,1/2]$ in this case and $\tilde{f}(\mathcal{M}_6) = \tilde{f}(\mathcal{M}_7) = \mathcal{M}_3$. The alphabet becomes

$$\tilde{\mathcal{A}} = \{1,2,3,4,5,63,73\}\ ,$$

and the zetafunction is

$$\frac{1}{\zeta_{A_1}} = 1-t_1-t_2-t_3-t_4-t_5-t_{63}-t_{73} = 1-5/\Lambda-2/\Lambda^2\ .$$

The other zetafunction from A_2 is now

$$\frac{1}{\zeta_{A_2}} = \frac{1-t_1+t_2+t_3-t_4-t_5-t_{63}-t_{73}}{1+t_3} = \frac{1-1/\Lambda-2/\Lambda^2}{1+1/\Lambda} = 1-2/\Lambda\ .$$

The Product becomes

$$1/\zeta(1,0) = (1-2/\Lambda)(1-5/\Lambda-2/\Lambda^2)\ .$$

In the case $\Lambda = 3+2\sqrt{2}$ we get:

$$\tilde{f}(\mathcal{M}_6) = \tilde{f}(\mathcal{M}_7) = \mathcal{M}_3 \cup \mathcal{M}_5 \cup \mathcal{M}_7$$

The symbolic dynamics can be described in the alphabet

$$\tilde{\mathcal{A}} = \{1,2,4,67^k3,67^k5,7^k3,7^k5,\overline{7}\}, \qquad k=0,1,2,\ldots$$

The zetafunction from A_1 becomes

$$\begin{aligned}
\frac{1}{\zeta_{A_1}} &= \left(1-t_1-t_2-t_4-\frac{t_3+t_5+t_{63}+t_{65}}{1-t_7}\right)(1-t_7) \\
&= 1-t_1-t_2-t_3-t_4-t_5-t_7-t_{63}-t_{65}+t_7(t_1+t_2+t_4) \\
&= 1-6/\Lambda+1/\Lambda^2
\end{aligned}$$

For ζ_{A_2} we get

$$\begin{aligned}
\frac{1}{\zeta_{A_2}} &= \frac{1-t_1+t_2+t_3-t_4-t_5+t_7-t_{63}+t_{65}-t_7(t_1-t_2+t_4)}{1+t_3} \\
&= \frac{1-1/\Lambda^2}{1+1/\Lambda} = 1-1/\Lambda\ .
\end{aligned}$$

The value $\Lambda = 1/2(7+\sqrt{33})$ determines properties:

$$\tilde{f}(\mathcal{M}_6) = \tilde{f}(\mathcal{M}_7) = \mathcal{M}_3 \cup \mathcal{M}_5 \cup \mathcal{M}_7 \cup \mathcal{M}_6 \cup \mathcal{M}_4\ .$$

This gives an alphabet like:

$$\tilde{\mathcal{A}} = \{1,2,[6,7]^k4,[6,7]^k5,[6,7]^k3,\overline{6},\overline{7},\overline{67},\ldots\}\ ,$$

and the zetafunction in the fundamental cell becomes:

$$
\begin{aligned}
\frac{1}{\zeta_{A_1}} &= \left(1 - t_1 - t_2 - \frac{t_3 + t_4 + t_5}{1 - t_6 - t_7}\right)(1 - t_6 - t_7) \\
&= 1 - t_1 - t_2 - t_3 - t_4 - t_5 - t_6 - t_7 + (t_1 + t_2)(t_6 + t_7) \\
&= 1 - 7/\Lambda + 4/\Lambda^2 \\
\frac{1}{\zeta_{A_2}} &= \frac{1 - t_1 + t_2 + t_3 - t_4 - t_5 + t_6 + t_7 - (t_1 - t_2)(t_6 + t_7)}{1 + t_3} \\
&= \frac{1 + 1/\Lambda}{1 + 1/\Lambda} = 1 \ .
\end{aligned}
$$

Once again in agreement with the result from the last section:

$$1/\zeta(1,0) = 1 - 7/\Lambda + 4/\Lambda^2 \ .$$

3.2 The Case $\Lambda = 5$

If the dynamics in the full map of this particular case is hairy, the reduced map in the fundamental cell is very simple. We use the partion of the interval $[0, 1/2]$ as given on fig. 3. This time however the dynamics are completely described in the simple alphabet $\tilde{\mathcal{A}} = \{1, 2, 3, 4, 5\}$, which yields the simple zetafunctions

$$
\begin{aligned}
\frac{1}{\zeta_{A_1}} &= 1 - t_1 - t_2 - t_3 - t_4 - t_5 = 1 - 5/\Lambda \ , \\
\frac{1}{\zeta_{A_2}} &= \frac{1 - t_1 + t_2 + t_3 - t_4 - t_5}{1 + t_3} = \frac{1 - 1/\Lambda}{1 + 1/\Lambda} \ , \\
\frac{1}{\zeta(1,0)} &= \frac{1}{\zeta_{A_1}} \frac{1}{\zeta_{A_2}} = \frac{1 - 6/\Lambda + 5/\Lambda^2}{1 + 1/\Lambda} \ .
\end{aligned}
\tag{7}
$$

There still is a nonvanishing pole in eq. 7 .

4 Numerical determination of diffusion constants

In this section we present the data from a numerical IDL-procedure.
The data are obtained by iterating a number, N, of initial points uniformly distributed in the interval $[-1/2, 1/2]$, i times. In order to compare the effect of iterating for long or short times, the product $N \times i$ is kept constant, as this coresponds to (aproximately) the same CPU-time.
The recorded data consists of calculations of

$$\mathrm{dat}_n = \frac{\langle (x - \langle x \rangle)^2 \rangle}{2}$$

The diffusion constant D is obtained from the formula

$$\langle x_n^2 \rangle \simeq 2Dn \ ,$$

as the slope (see fig. 5 for an example). To avoid transients of the initial distribution, the first 2 iterations in the dataset are discardet. The results displayed in table 2 are the mean from 50 runs.

It seems that the predicted results all are close to the values obtained numerically and they are within the uncertainty of the numerical values. It pays better to make short iterations on many trajectories in all cases – the uncertainty is improved allmost by factor 10, when the number of trajectories are increased in the above way.

Table 1: Exact Results

Exact values	$\Lambda = 4$	$\Lambda = 5$	$\Lambda = 3 + 2\sqrt{2}$
D	0.10	0.40	$0.353553\ldots$

Table 2: Numerical Results

N:	i:	$\Lambda = 4$	$\Lambda = 5$	$\Lambda = 3 + 2\sqrt{2}$
10000	10	0.10021 ± 0.00114	0.39878 ± 0.00394	0.35448 ± 0.00355
1000	100	0.09941 ± 0.00489	0.39723 ± 0.0213	0.35657 ± 0.0191
100	1000	0.09605 ± 0.00733	0.38390 ± 0.0279	0.35389 ± 0.0267

4.1 Conserning Long iterations

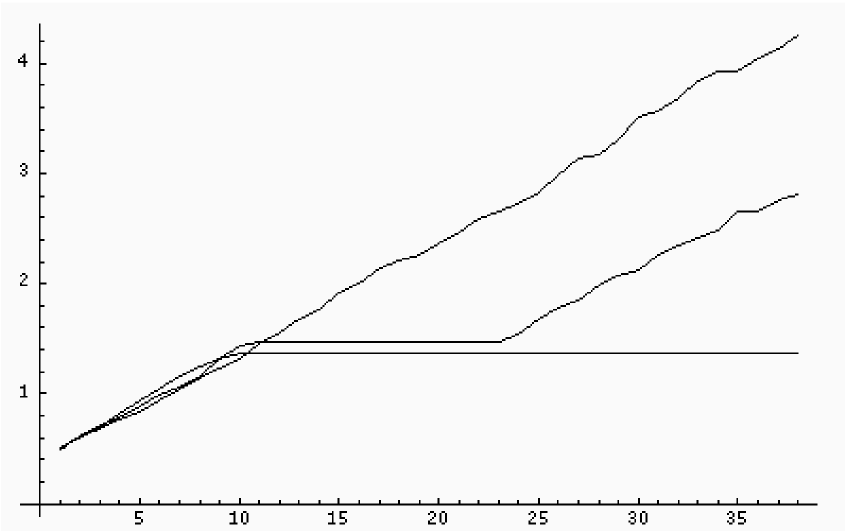

Figure 4: Diffenent dataset obtained with $N = 1000$ with different added noise $\Lambda = 4$. Top: with noise on 7^{th} decimal place. Middle: noise on 13^{th} decimal. Bottom: no noise.

When doing numerical simulations, a problem arises when we want to iterate for long times. For values of $\Lambda = 4, 6, 8, \ldots$ any trajectory will ultimately end at a fixed point with $x = 0.00000, \pm 0.50000, \pm 1.0000, \ldots$. If the system then is perturbed slightly the trajectories wander on a long excursion before they settle at new fixed points. For $\Lambda = 4$ the maksimal iteration length is close to 10 iterations. This is illustrated on Fig. 4. There were no effects of this kind in the runs with $\Lambda = 5$ as illustrated in fig. 5.

The remedy for these effects was to add a tiny noise term on the last decimal place (the 7^{th} in IDL double precicion), that is after each iteration a random number in the interval $[-5 \cdot 10^{-8}, 5 \cdot 10^{-8}]$ is added. Interval is chosen such that the mean after many iterations is zero. The makroscopic effect in terms the computed averedges of this is not noticeble on the level of presision of these simulations as seen on fig. 5. When iterating on a computer, any initial

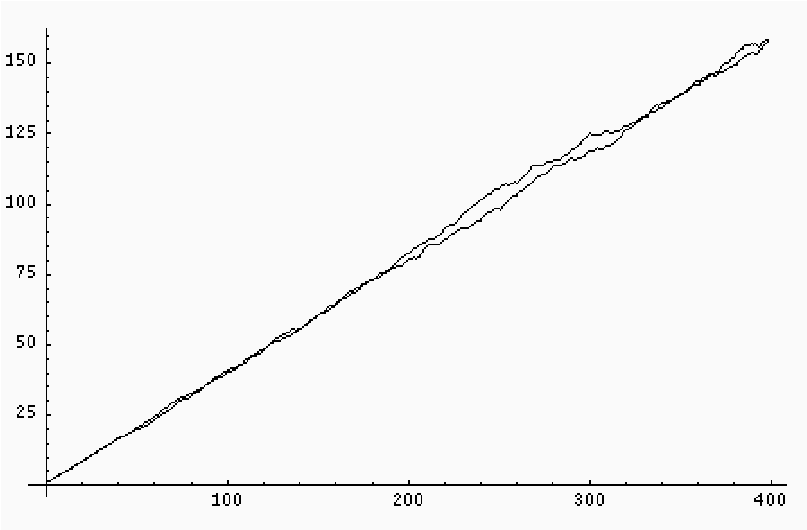

Figure 5: Two dataset with and without noise for the value $\Lambda = 5$. $N = 1000$.

point, will be given with only a finite number of decimals I belive this to be the reason for this behaviour.