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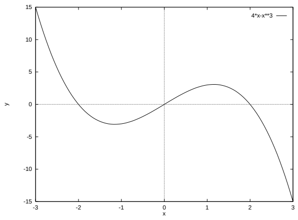
Project for the spring course "Classical and Quantum Chaos"

The Spectrum of the Liouville and the Fokker-Planck equation at the Pitchfork bifurcation



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The exam project

My task was to understand a recent article of Gaspard et al. on the topics descibed below:

Abstract

The Liouville equation describing the evolution of probabillity densities is studied at the pitchfork bifurcation by considering the spectrum, the eigenfunctions and the dual eigenfunctions of the corresponding Liouville operator. At criticality the system exhibits a continuos spectrum and above the Liouville operator decomposes in Jordan-blocks. Noise is added and the features of the Fokker-Planck equation correspond with the Liouville equation except from the disappearance of the continuos spectrum at criticality.

Keywords:
Liouville equation, spectral theory, pitchfork bifurcation, Fokker-Planck equation

Contents

1 Introduction

During the course "Classical and Quantum Chaos" ref. [1] we have been studying the spectrum of various evolution operators. The way we did this was by using an alternative - but computational efficient approach - namely via periodic orbits. In the following text however we shall see the conventional way of finding spectral features. The physical proces we are going to study is the so-called pitchfork-bifurcation which is one of the simplest bifurcations. This bifurcation (see fig.1 later) is universal for all anayltical 1-d vector fields, ie. in suitable coordinates we can rewrite the bifurcation as the pitchfork. The operators we want to study at this bifurcation are the statistical operators of Liouville and Fokker-Planck, which describe the evolution of probabillity densities in a dynamical system without resp. with noise. It sounds as a formidable task. Can one really follow the spectrum etc. during the bifurcation? Yes, this is done in ref. [2] quite pedagogically, and we will try to redo in the following.

2 The Liouville equation

Short introduction to the Liouville-equation

We consider a dynamical system of the form

$$\dot{\vec{x}} = F(\vec{x};\mu). \tag{1}$$

If one writes a continuity equation for a density one finds

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\vec{v}\rho) = 0. \tag{2}$$

Let us assume that this velocity is given by the dynamical system; ie. $\vec{v} = \dot{\vec{x}} = F(\vec{x};\mu)$. Therefore the continuity equation is equivalent to

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\vec{F}(x;\mu)\cdot\rho) \equiv \hat{L}\rho(x). \tag{3}$$

This partial differential equation is called the Liouville equation and the operator $\hat{L}$ is called the Liouville operator. Thus the Liouville operator $\hat{L}$ and its adjoint $\hat{L}^\dagger$ is given by $-\nabla\cdot(F(\vec{x};\mu)\cdot)$ respectively $F(\vec{x};\mu)\cdot\nabla$ (since $\frac{\nabla}{i}$ is self-adjoint).

We write a formal solution to the dynamical system as $\dot{\vec{x}} = \vec{f}^t(x;\mu)$ and one then sees that

$$\rho(\vec{x},t) = \int dy \rho_0(\vec{y})\delta(\vec{x} - \vec{f}^t(x;\mu)). \tag{4}$$

The kernel of this integral is the operator we have been studying in the CATS - chaos course. In the following however we are more intersted in the following (formal) representation of $\rho(\vec{x},t)$:

$$\rho_t = exp(t\hat{L})\rho_0. \tag{5}$$

Spectral decomposition

We now wish to consider the spectrum of the Liouville operator. The approach of Predrag et al. where one ends up with trace-formulas expressed as sums over periodic orbits is worked out by using the kernel of the integral above (4). Here in this section however we explicitely work with eigenfunctions or -distributions. For simplicity at this point we assume a discrete set of eigenvalues of $\hat{L} : \{s_n\}$ and eigenfunctions $\{\phi_n\}$. The dual vectors - so-called left eigenvectors - are denoted $\tilde{\phi_n}$.

Since formally

$$\rho_t = exp(t\hat{L})\rho_0, \tag{6}$$

we get

$$\rho(\vec{x},t) = \sum_{n=0}^{\infty} < \tilde{\phi_n}, \rho_0 > \phi_n(\vec{x})e^{s_n t} \tag{7}$$

after using the closure relation (as in QM).

In the same way one could investigate averages of observables A:

$$\begin{aligned} < A >_t &= \int A(\vec{x})\rho(\vec{x},t)d\vec{x} =< A, exp(t\hat{L})\rho_0 > \\ &= < A, \sum_{n=0}^{\infty} < \tilde{\phi_n}, \rho_0 > \phi_n(\vec{x})e^{s_n t} > \\ &= \sum_{n=0}^{\infty} e^{s_n t} < A, \phi_n >< \tilde{\phi_n}, \rho_0 > . \end{aligned} \tag{8}$$

We remark that this is not true if $\hat{L}$ admits a Jordan block structure. We shall return to this point later and redo the above formal manipulations in that particular case.

Since we are discussing infinite dimensional operators a careful treatment of the spaces of initial distributions ρ_0 and observables A is necessary. We refer to ref. [2] where it is shown that the space of observables J_A can be taken as the entire functions contrary to the space of initial densities J_{ρ_0} must be taken as Scwartz distributions.

3 Plan of Battle

Let us try and find the spectrum, the right - and the left eigenvectors in the following cases:

Deterministic : Liouville equation
linear $\dot{x} = \mu x$
non-linear $\dot{x} = \mu x - x^3$
Stochastic : Fokker-Planck
linear $\dot{x} = \mu x + W(t)$
non-linear $\dot{x} = \mu x - x^3 + W(t)$,
where the sub-critical , critical and super-critical values of μ are considered.

The methods are quite simple: Taylor-expansions of composed functions and use of delta-functions and their derivatives. As an illustration of these methods we start with a very simple system:

1-d vector field , linear case:

The dynamical system $\dot{x} = \mu x$ has the following flow-function:

$$x = f^t(x_0) = x_0 e^{\mu t}. \tag{9}$$

Therefore the integral representation of the evolution operator is

$$\rho(x,t) = \int_{-\infty}^{+\infty} \rho_0(y)\delta(x - ye^{\mu t})dy. \tag{10}$$

So for μ negative we can Taylor expand the delta-function.

On the other hand the lio equation is

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\vec{F}\rho) = -\mu \frac{\partial (x\rho)}{\partial x}. \tag{11}$$

If this equation is solved (eg. by the methods of characteristics) for ρ one finds

$$\rho(x,t) = e^{-\mu t}\rho_0(xe^{-\mu t}). \tag{12}$$

In this case we see that a Taylor expansion of ρ_0 works for μ positive.

One gets

$$\rho(x,t) = \sum_{n=0}^{\infty} e^{\mu n t}[\int_{-\infty}^{+\infty} \frac{y^n}{n!}\rho_0(y)dy](-1)^n \delta^{(n)}(x) \tag{13}$$

for $\mu < 0$ and

$$\rho(x,t) = \sum_{n=0}^{\infty} e^{\mu(n+1)t}[\int_{-\infty}^{+\infty} (-1)^n \delta^{(n)}(y)\rho_0(y)dy]\frac{x^n}{n!} \tag{14}$$

for $\mu > 0$.

Here in the first case the derivatives of the delta-function comes from its Taylor expansion, where in the last case the action of these derivatives (use partial integration) exactly reproduces the Taylor - coefficients of the expansion

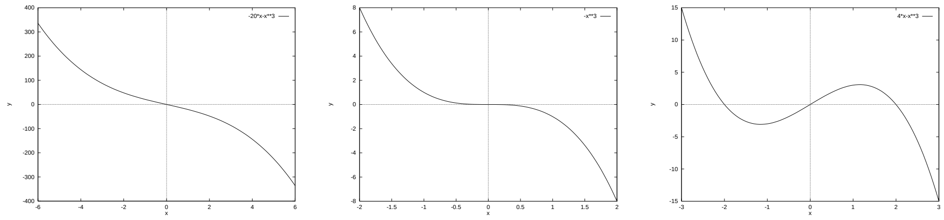

Figure 1: The unfolding of the bifurcation as μ varies

of ρ_0. Now it is straightforward to read off the eigenvalues, eigendistributions and dual eigendistributions. We find:

$$
\begin{aligned}
\mu < 0 : s_n &= n\mu = -n|\mu|, n = 0, 1, 2, ... \\
\phi_n(x) &= (-1)^n \delta^{(n)}(x), \\
\tilde{\phi_n}(x) &= \frac{x^n}{n!}.
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
\mu > 0 : s_n &= -(n+1)|\mu|, n = 0, 1, 2, ... \\
\phi_n(x) &= \frac{x^n}{n!}, \\
\tilde{\phi_n}(x) &= (-1)^n \delta^{(n)}(x).
\end{aligned}
\tag{16}
$$

$$\tag{17}$$

Do we only have a discrete set of eigenvalues?
Let us consider the case $\mu > 0$. One easily sees that x^α $(Re(\alpha) > 0)$ also is an eigenfunction with eigenvalue $s = -\mu(\alpha + 1)$. But this eigenfunction is not analytic at 0. Anyway if the functional spaces of the eigenfunctions are enlarged to nonsmooth test functions we must conclude that there is a continuous spectrum. The discrete set of eigenvalues is considered as emerging from a continuous set lying in the complex plane. That is if you consider your Liouville operator on a larger space you may expect many more eigenvalues than just a discrete set (Pollicott-Ruelle resonances). See ref. [2] for references.

4 The Pitchfork

Let us consider the one-dimensional dynamical system describing this bifurcation

$$\frac{dx}{dt} = \mu x - x^3. \tag{18}$$

Here μ is the external parameter which controls the bifurcation. See fig. 1. For $\mu < 0$. 0 is the only fixpoint and it is attracting. For $\mu > 0$ two new fixpoints $\sqrt{\mu}$ and $-\sqrt{\mu}$ appear. They are attracting. 0 is still a fixpoint but now a repelling one. One easily finds the flow-function $\vec{x}(t) = f^t(\vec{x}_0, \mu)$ by solving the above first order differential equation.

5 Sub-critical regime

Here the external parameter is $\mu < 0$:

In ref. [2] they introduce a new variable

$$\xi = \frac{xe^{-|\mu|t}}{\sqrt{x^2 + |\mu|}} \tag{19}$$

which they use for taylor expansion of A. Omitting the details we find from ref. [2]

$$\begin{aligned}
s_n &= -n\,|mu|\,, \qquad n = 0, 1, 2, ..., \\
\phi_n &= \partial_\xi^n \delta \left[x - \frac{|\mu|^{1/2}\xi}{\sqrt{1-\xi^2}} \right] \Bigg|_{\xi=0} \\
&= \sum_k (-1)^k \delta^{(k)}(x) \frac{n!(n-2)!!}{k!(k-2)!!} \frac{\sqrt{|\mu|^k}}{2^{(n-k)/2}[\frac{n-k}{2}]!}, \\
&\qquad \text{with } k = 2, 4, ... \text{ for } n \text{ even and } k = 1, 3, ... \text{ for } n \text{ odd .} \\
\tilde{\phi_n}(x) &= \frac{1}{n!}[\frac{x}{\sqrt{x^2 + |\mu|}}]^n.
\end{aligned} \tag{20}$$

Let us instead investigate in more detail the general case in 1-d for any holomorphic vector field $\dot{x} = F(x)$ containing a single attracting fixpoint at the origin (which is the case of $\mu < 0$) such that $F(0) = 0$ and $F^{(1)}(0) = \lambda = -|\lambda|$. By a theorem of Poincaré there exists a biholomorphic transformation $\xi = g(x)$ such that the dynamical system becomes

$$\dot{\xi} = \lambda \xi. \tag{21}$$

Furthermore g is defined in the maximal disk not containing any of the complex fixpoints.

The ξ-flow is exponential

$$\xi(t) = g^t(\xi) = e^{\lambda t}\xi \tag{22}$$

and hence

$$\begin{aligned}
< A >_t &= \int A(f^t(x))\rho_0(x)dx = \int A(g^t(\xi))\rho_0(\xi)d\xi \\
&= \int A(e^{\lambda t}\xi(\xi))\rho_0(\xi)d\xi \\
&= \int \sum_n \frac{e^{n\lambda t}\xi^n}{n!} A^{(n)}(0)\rho_0(\xi)d\xi \\
&= \sum_n e^{n\lambda t} A^{(n)}(0) \int \frac{\xi^n}{n!}\rho_0(\xi)d\xi
\end{aligned} \tag{23}$$

from which we read off

$$s_n = n\lambda = -n|\lambda|, \qquad \phi_n = \frac{\partial^(n)}{\partial \xi^n}\delta(x - g^{-1}(\xi))|_{\xi=0}, \qquad \tilde{\phi}_n = \frac{g(x)^n}{n!}. \tag{24}$$

So the spectrum is unchanged as in the linear model. However the eigenvectors now contain the non-linearities. These considerations can be generalized to higher dimensions, see ref. [2] for the result.

Finally we remark that as $|\mu| \to \infty$ the right eigenvectors tend to the right eigenvectors of the linear case (the distributions get attracted to 0 and end up feeling nothing but the linear part of the vector field). This is seen as follows : only the highest term in the expression for ϕ_n, ie. the k=n term, contributes in the limit of $|\mu| \to \infty$. However for $t \to -\infty$ the distributions feel the nonlinear part of the field and we have no correspondence to the linear case, in fract $\tilde{\phi}_n(x) \to \pm 1$.

6 Critical regime

Here we put $\mu = 0$ and thus there is no linear term in the vector field.

The dynamical system therefore is $\dot{x} = -x^3$ and the corresponding flow-function is easily found to be

$$x = f^t(x_0) = \frac{x_0}{\sqrt{1 + 2x_0^2 t}}. \tag{25}$$

Hence $x \sim 1/\sqrt{t}$ for $t \to \infty$. So this is not an exponential convergence to a fixpoint. It is much slower. This phenomena is called critical slowing down.

We remark the following: The first derivative of the vectorfield vanishes! Therefore the above calculation for a general 1-d vector field no longer is valid. Indeed we will see this by the presence of a continuos spectrum.

The average becomes ?refer?

$$\begin{aligned} < A >_t &= \int dx \rho_0(x) A(\frac{x}{\sqrt{1+2x^2 t}}) \\ &= \int dx \rho_0(x) \sum_{n=0}^{\infty} \frac{\frac{x^n}{(1+2x^2t)^{n/2}}}{n!} A^{(n)}(0). \end{aligned} \tag{26}$$

We see that we don't have any exponential decay factors yet. If one uses the properties of the Gamma function one gets an integral of these decay factors:

$$\frac{x^n}{(1+2x^2t)^{n/2}} = \frac{1}{2\Gamma(n/2)} \int_{-\infty}^{0} ds e^{st} [-\frac{s}{2}]^{n/2-1} [x/|x|]^n e^{\frac{s}{2x^2}}. \tag{27}$$

One then finds the following spectral decomposition:

$$< A >_t = < A, \phi_0 >< \tilde{\phi_0}, \rho_0 > + \int_{-\infty}^{0} ds e^{st} [< A, \phi_{s+} >< \tilde{\phi_{s+}}, \rho_0 > + < A, \phi_{s-} >< \tilde{\phi_{s-}}, \rho_0 >], \tag{28}$$

where for the eigenvalue s=0 we have

$$\phi_0 = \delta(x) \text{ and } \tilde{\phi_0} = 1. \tag{29}$$

So we simply have that the invariant probability density located at 0 stays at this fixpoint.

The continuous spectrum is seen to be doubly degenerate, since for a given $s < 0$ both ϕ_{s+} and ϕ_{s-} are eigenvectors.

The eigenvectors of even parity are:

$$\phi_{s+} = \sum_{neven>0}^{\infty} \frac{(-1)^n}{2n!\Gamma(n/2)}[-\frac{s}{2}]^{n/2-1}\delta^n(x)\tilde{\phi_{s+}} = e^{\frac{s}{2x^2}}, s<0 \tag{30}$$

The eigenvectors of odd parity are:

$$\phi_{s-} = \sum_{nodd>0}^{\infty} \frac{(-1)^n}{2n!\Gamma(n/2)}[-\frac{s}{2}]^{n/2-1}\delta^n(x)\tilde{\phi_{s-}} = sgn(x)e^{\frac{s}{2x^2}}, s<0. \tag{31}$$

<u>An obvious eigenfunction possesing a singularity</u>

By insertion one easily sees that $\hat{L}\tilde{\phi_{s\pm}} = s\tilde{\phi_{s\pm}}$, since $\hat{L} = \vec{F}\cdot\nabla = -x^3\partial_x$. In the same way one finds that $\chi_s(x) = \frac{1}{x^3}exp[-\frac{s}{2x^2}]$ is an eigenfunction of $\hat{L} = \partial_x(x^3\cdot)$. But χ_s is not defined at the origin since it has a singularity there. What is the relation between this eigenfunction and the distributions ϕ_{s+} and ϕ_{s-}? The overall argument for a much simpler situation is like this:

$$<\delta, A> = \int_{-\infty}^{\infty} dx\delta(x)A(x) = A(0) = \frac{1}{2\pi i}\oint dz\frac{A(z)}{z} \equiv < \frac{1}{z}, A>_{contour} . \tag{32}$$

That is, the action of a δ-function on test-functions can also be represented by the contour integral of 1/z, the Cauchy-kernel. Of course we must realize that 1/z has a singularity at 0, even though that does not pose any problems, since we circle around it.

In short ref. [2] simply finds out of how to represent ϕ_s's action on observables in terms of some complex function. This turns out to be χ_s with a more complicated contour than just a trip around the origin as for 1/z. See in ref. [2] the appendix B fig. 8.(b) for details; all one should know is some residual calculus. From ref. [2] (B5) we get

$$\begin{aligned} <A, \phi_{s\pm}> &= \frac{i}{2\pi}\int_{\Gamma'}\frac{dx}{x^3}exp[-\frac{s}{2x_0^2}]\frac{1}{2}(A(x)\pm A(-x)) \\ &= <\frac{A\pm SA}{2}, \chi_s>_{\Gamma'}, \end{aligned} \tag{33}$$

where $SA(x) = A(-x)$ and Γ' is the contour described in app.B fig.8(b) of ref. [2] and the last equality should be regarded as a definition. This was the connection sought. We may therefore say that modulo representation χ_s is $\phi_{s+} + \phi_{s-}$.

Let us use the ref. [2] calculations to show (B5). Consider the dual vector called

$$\int_{-\infty}^{0} ds\tilde{\phi_{s\pm}}(x_0) < \frac{A\pm SA}{2}, \chi_s >_{\Gamma'} + A(0) \tag{34}$$

$$= \int_{-\infty}^{0} ds\tilde{\phi}_s(x_0) < \frac{A\pm SA}{2}, \chi_s >_{\Gamma'} = ... = A(x_0)!, \tag{35}$$

since

$$\int_{-\infty}^{0} dsexp[\frac{s}{2x_0^2}] < \frac{A\pm SA}{2}, \chi_s >_{\Gamma'} = \frac{1}{2}(A(x)\pm A(-x) - (1\pm 1)A(0)), \tag{36}$$

with $x = |x_0|$ and in fact $\tilde{\phi}_{s-} = sgn(x)e^{\frac{s}{2x^2}}$. $A(x)$ is indeed dual since it acts on the right eigenvectors, the ϕ's. Now we use orthonormality to infer that (B5) actually holds. We also see an expansion of this arbitrary dual vector $A(x)$ in terms of the eigenfunctions. This gives us completeness.

As a final remark we calculate the correspondence between the dual eigenvectors before the bifurcation, sub-critical, and the dual vectors at the bifurcation. We have with $s = \mu n$ constant and n even during the limit $\mu \to 0-$ that

$$n! \cdot \tilde{\phi_n}(x) = [\frac{x}{\sqrt{x^2 + |s|/n}}]^n \simeq (\frac{1}{\sqrt{1 - \frac{s}{nx^2}}})^n \simeq exp[\frac{s}{2x^2}] = \tilde{\phi_{s+}}. \tag{37}$$

So we can in fact approximate any of the continuos critical eigenvalues by the discrete sub-critical ones, which Gaspard et al. also noted in their earlier paper ref. [3]. See (fig. 1) in ref. [2] for this accumulation of eigenvalues.

7 Super-critical regime

Super-critical case $\mu > 0$:

In this section we wish to make it plausible that the Liouville operator no longer is diagonalized but only admits a Jordan block decomposition. For allmost all the calculational details we refer to ref. [2].

First how does the average $< A >_t$ behave when we have a Jordan block? Let us assume that $\hat{L} = x + yT$, where $T^m = 0$ for some m (natural number). Thus $\hat{L}$ consists of an identity part and an idempotent part. Since the identity part commutes with everything including T we will have no problems in exponentiating $\hat{L}$. We find

$$< A >_t = < A, e^{t\hat{L}}\rho_0 > = \sum_n < A, e^{t\hat{L}}\phi_n >< \tilde{\phi_n}, \rho_0 > . \tag{38}$$

Now

$$e^{t\hat{L}} = e^{tx}(1 + ytT + ... + \frac{(yt)^{m-1}}{m-1!}T^{m-1}), \tag{39}$$

since the taylor expansion of e^{yT} terminates at the m-1'st order. The result becomes

$< A >_t = \sum_n < \tilde{\phi_n}, \rho_0 > (< A, \phi_n > +yt < A, T\phi_n >$
$+\frac{(yt)^2}{2} < A, T^2\phi_n > +... + \frac{(yt)^{m-1}}{m-1!} < A, T^{m-1}\phi_n >)$,
where $m = m_n$. Hence the time evolution becomes slower because of the presence of $t^m e^{-n\mu t}$-terms in the average. This is a sign of critical slowing down.

We now take the limit $t \to \infty$. In this way we obtain the leading term in $< A >_t$ since it contains the leading eigenvalue as $e^{s_n t}$. This term is then subtracted from the original average to get the subleading term and so on.

In the domain 1+ - here we use ref. [2]'s conventions; ie. between 0 and the fixpoint most to the right - we find

$$< A >_t = \int_{1+} dy A(\frac{y\sqrt{\mu}}{\sqrt{y^2 + (\mu - y^2)e^{-2\mu t}}}) \to A(\sqrt{\mu}) \int_{1+} \rho_0(y)dy. \tag{40}$$

This corresponds to the eigenvalue $s_0 = 0, 1 = e^{s_0 t}$ and the eigenvectors defined by

$$< A, \phi >= A(\sqrt{\mu}) \tag{41}$$

and

$$< \tilde{\phi}, \rho_0 >= \int_{1+} \rho_0(y) dy. \tag{42}$$

The subleading term gives the eigenvalue $s_1 = -\mu$ and eigenvectors defined by

$$< \tilde{\phi}, \rho_0 >= \rho_0(0) \tag{43}$$

and

$$< A, \phi >= \int_{1+} \frac{\mu^{3/2}}{(\mu - x^2)^{3/2}} (A(x) - A(\sqrt{\mu})) dx. \tag{44}$$

Then the block appears...: Just as above we subtract the subleading terms from the average to obtain
$< A >_t -$ leading term $-$ subleading term $= exp(-2\mu t)$
$[\rho_0'(0) \int dx \frac{\mu^2 x}{(\mu - x^2)^2} [A(x) - A(\sqrt{\mu}) - \frac{x^2 - \mu}{2\sqrt{\mu}} A'(\sqrt{\mu})] - \frac{\sqrt{\mu}}{2} A'(\sqrt{\mu}) \int dy \frac{\mu - y^2}{y^2} [\rho_0(y) - [\frac{\mu}{\mu - y^2}]^{3/2} \rho_0(0) - \frac{\mu y}{\mu - y^2} \rho_0'(0) - t \frac{\mu^{5/2}}{2} A'(\sqrt{\mu}) \rho_0'(0)] + O(e^{-3\mu t}) + O(te^{-4\mu t})$.

Thus the eigenvalue corresponds to $s_2 = -2\mu$. We see that the dimension of the Jordan block is 2. Let us call the eigenvectors ϕ_a and $\phi_b = T\phi_a$. Then the last term has the form

$< A, T\phi_a >< \tilde{\phi}_a, \rho_0 >$. Here we can identify (up to a constant) the b-vector and the a-dual-vector. The two first terms however are of the form

$$< \tilde{\phi}_a, \rho_0 >< \phi_a, A > + < \tilde{\phi}_b, \rho_0 >< \phi_b, A > \tag{45}$$

and we can then find the b-dual-vector and the a-vector: $< \phi_a, A >= \int_{1+} dx \frac{\mu^2 x}{(\mu - x^2)^2} [A(x) - A(\sqrt{\mu}) - \frac{x^2 - \mu}{2\sqrt{\mu}} A'(\sqrt{\mu})]$
$< \phi_b, A >= \frac{\sqrt{\mu}}{2} A'(\sqrt{\mu})$
$< \tilde{\phi}_a, \rho_0 >= \rho_0'(0)$ $< \tilde{\phi}_b, \rho_0 >= \int_{1+} dy \frac{-\mu + y^2}{y^2} [\rho_0(y) - (\frac{\mu}{\mu - y^2})^{3/2} \rho_0(0) - \frac{\mu y}{\mu - y^2} \rho_0'(0)]$.
In the same way one finds a block structure in the domain to the left of 0 in the domain 1-. It turns out however that the development of the spectral decomposition of the domains facing $\pm\infty$ runs exactly as in the subcritical regime. We refer to ref. [2] for the results there. The eigenvalue $-2|\mu|$ appears in all 4 domains. Taking the degeneracy of the domains 1$\pm$ into consideration one has at most a dimension of 6=1+2+2+1 related to this eigenvalue for a global distribution. But then the matching of this distribution at the 3 fixpoints reduces the dimension to 6-3=3. Now consider the s_0=0 eigenvalue. We will describe the matching in this case. The point is that we have matching at the repelling fixpoint 0. We see from above that $\phi_0^{1+} = \delta(x - \sqrt{\mu})$. And it is not hard to show that $\phi_0^{1-} = \delta(x + \sqrt{\mu})$, since all that happens in the $t \to \infty$ limit of (3.47) of ref. [2] is that we integrate over the negative domain 1-. And now we see it: the δ's do indeed match

at 0. Then one finds the following dimension of the eigenspace related to s_0=0: 1+1+1+1-2=2. Thus pictorially we have 2 δ-peaks over the attractiong fixpoints, each with some factor in front.

We notice that it is not typical that an operator has degenerate eigenvalues, therefore it will not be typical that it has a Jordan block decomposition (since we have degeneracies connected with every Jordan block). Hence we will in general have exponential decay of our observables.

Finally some words about degeneracies: Since the stabillity exponents at each of the fixpoints actually are in ratio $\lambda^{(\pm)} = -2\lambda^{(0)}$ the spectrum turns out be degenerate. This is because these exponents as we have seen explicitely for the pitchfork-bifurcation determine the spectrum given by expressions of the form $-n|\lambda|$. Thus clearly for n even the eigendistributions in the 2$\pm$- domains at level $\frac{n}{2}$ have the same eigenvalues as the n-level eigendistributions in the 1$\pm$ -domains.

8 The Fokker-Planck equation

We add noise to the dynamical system

$$\dot{\vec{x}} = F(\vec{x};\mu) + \vec{W}. \tag{46}$$

Here the isotropic gaussian white-noise term fulfills

$$< \vec{W(t)} >= 0 \text{ and } < \vec{W(t)}\vec{W(t')} >= 2D|\delta(t-t'), \tag{47}$$

where | is the identity matrix.

This defines a diffusion process - Brownian motion - and its probability density satisfies the following modified continuity equation - the Fokker-Planck equation ref. [4]:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\vec{F}\rho) = D\Delta\rho. \tag{48}$$

Similarly one defines a Fokker-Planck operator

$$\hat{L_{FP}} = -\nabla\cdot(\vec{F}\rho) + D\Delta\rho. \tag{49}$$

This operator also admits a spectral decomposition and the Liouville operator is recovered in the noise-less limit D $\to$ 0.

The Fokker-Planck equation is related to the Schroedinger equation in the following way:

First we write the vector field as minus the gradient of some potential:

$$\vec{F}(x) = -\partial_x U. \tag{50}$$

Then a staionary solution to the Fokker-Planck equation is given by

$$\rho = N_0 exp[-\frac{U}{D}]. \tag{51}$$

Varying the normalization constant, ie. trying an ansatz of the form $\rho = exp[-\frac{U}{2D}]f$, we get a Schoedinger-like equation for f:

$$\partial_t f = -\hat{H}f, \tag{52}$$

where $\hat{H} = -D\partial_x^2 + \Phi$, with

$$\Phi = -\frac{1}{2}\partial_x^2 U + \frac{1}{4D}(\partial_x U)^2 = \frac{1}{2}\nabla \cdot \vec{F} + \frac{1}{4D}F^2. \tag{53}$$

The relation to the original problem formulated in terms of ρ is as follows: For typical steep potentials the eigenvalues of $\hat{H}$ are discrete and for solutions of the eigenvalue problem:

$$\hat{H}\psi_m = -\sigma_m \psi_m \tag{54}$$

one has equivalently:

$$\hat{L}_{FP}\phi_m = \sigma_m \phi_m, \text{ with } \phi_m = exp[-\frac{U}{2D}]\psi_m, \tag{55}$$

$$\hat{L}_{FP}^{\dagger}\phi_m = \sigma_m \tilde{\phi_m}, \quad \text{with } \tilde{\phi_m} = exp[+\frac{U}{2D}]\psi_m. \tag{56}$$

Since ϕ and $\tilde{\phi}$ are orthonormal we find the same expression as for the Liouville equation:

$$\rho(x,t) = \sum_{n=0}^{\infty} < \tilde{\phi_m}, \rho_0 > \phi_m(x) e^{\sigma_m t}. \tag{57}$$

8.1 Fokker-Planck: linear case

In this section we wish to see the correspondence with the linear system without noise.

With a vector field $F(x) = \mu x$ we get the corresponding "quantum"-potential

$$\Phi = \frac{1}{4D}F^2 + \frac{1}{2}\partial_x F = \frac{\mu^2}{4D}x^2 + \frac{\mu}{2}, \tag{58}$$

ie. like the harmonic oscillator. Putting $D = \frac{\hbar^2}{2m}$ one easily finds that the level-spacing is $\hbar\omega = |\mu|$. Now if one notices the sign difference between the $\hat{H}$ and $\hat{L}_{FP}$ eigenvalues one actually find after a little calculation the same eigenvalues as in the Liouville case:

$$\sigma_m = -|\mu|(m+1) \tag{59}$$

for $\mu > 0$,

$$\sigma_m = -|\mu| m \tag{60}$$

for $\mu < 0$, where m=0,1,2,3,... .
However the noise has smeared out the deterministic particles from distributions to regular functions:

For $\mu < 0$:

$$\phi_m = N_m H_m(\alpha x) exp[-\frac{|\mu|}{2D}x^2] \text{ and} \tag{61}$$

$$\tilde{\phi}_m = \tilde{N}_m H_m(\alpha x), \tag{62}$$

where for $\mu > 0$

$$\phi_m = \tilde{N}_m H_m(\alpha x) \text{ and} \tag{63}$$

$$\tilde{\phi}_m = N_m H_m(\alpha x) exp[-\frac{|\mu|}{2D} x^2], \tag{64}$$

with $\alpha = \sqrt{|\mu|/(2D)}$ (this $\alpha = (\frac{mK}{\hbar^2})^{1/4}$, where K is the spring constant (see any book in Quantum Mechanics) $= \frac{\mu^2}{2D}$ here) and the H_α's Hermite-polynomials of course coming from the harmonic oscillator.

Now these functions are not at all so different from the Liouville case. Using the normalisation $N_m = \frac{\alpha^{m+1}}{\sqrt{\pi}}$ and Rodrigues' formula for a representation of Hermite-polynomialswe we find for $\mu < 0$

$$\begin{aligned}
\phi_m &= \frac{\alpha^{m+1}}{\sqrt{\pi}}(-1)^m e^{(\alpha x)^2}(\frac{d^m}{d(\alpha x)^m} e^{-(\alpha x)^2}) e^{-(\alpha x)^2} \\
&= \frac{\alpha}{\sqrt{\pi}}(-1)^m \frac{d^m}{dx^m} e^{-(\alpha x)^2} \\
&= (-1)^m \frac{d^m}{dx^m}(\frac{\alpha}{\sqrt{\pi}} e^{-(\alpha x)^2}) \\
&\rightarrow (-1)^m \delta^m(x),
\end{aligned} \tag{65}$$

when $\alpha \rightarrow \infty$ which is the case when $D \rightarrow 0$. A similar analysis is also possible for the dual vectors, see ref. [2].

We therefore conclude that the spectral decomposition of the Liouville equation is recovered in the noise-less limit. If we are a little pedantic we have to extend the Fokker-Planck equation's smeared out distributions to actual distributions before one takes the limit!

8.2 Fokker-Planck: non-linear case

The pitchfork bifurcation and FP:

The Langevin equation becomes

$$\dot{x} = F(x) = \mu x - x^3 + W(t) \tag{66}$$

with the corresponding quantum-like potential

$$\Phi = \frac{1}{4D} F^2 + \frac{1}{2}\partial_x F = \frac{x^2}{4D}(x^2 - \mu)^2 + \frac{\mu}{2} - \frac{3}{2}x^2 \tag{67}$$

(as shown in ref. [2] fig.2): *** When $\mu < 0$ the potential has only one well. Then during the bifurcation actually three wells appear. Furthermore in the limit of no noise the levels of these wells tend to the same height and the walls become steeper and steeper.

One easily verifies that the noise-less limit corresponds to the limit $|\mu| \rightarrow \infty$ and $D = 1$ if one makes the following change of variables:

$$x \rightarrow D^{1/4}x, t \rightarrow D^{-1/2}t, \mu \rightarrow D^{1/2}\mu, \ \sigma_m \rightarrow D^{1/2}\sigma_m. \tag{68}$$

Now in ref. [2] they have numerically integrated the Fokker-Planck equation in the different regions of μ corresponding to different regions of D. See ref. [2] (fig.4) and (fig. 5) for graphical representation of the eigenfunctions.

The important thing is that they find (numerically) a correspondence with the Liouville equation spectrum below and above criticality. We simply refer to ref. [2] for the details (one only needs to think of the found eigenfunctions as Gaussian lumps on the real line and derivatives thereof, and then perform the limit to get the Liouville equation's δ's etc.).

However the spectrum ceases to be continuos when noise is added! The explanation of this phenomenon is just as before (see also ref. [2] around (2.29)) that if one adds noise the distributions get smeared out, ie. become much more regular. Thus the eigenfunctions belong to a much smaller vectorspace and therefore the spectrum shrinks. At criticality we had a continuous spectrum but after adding noise it becomes discrete. This is seen in ref. [2] fig.3 . Around $\mu = 0$ there is a region where one clearly sees the discrete eigenvalues. But then when $D \to 0$ this region shrinks (they found numerically) and the eigenvalues accumulate on the negative line at $\mu = 0$.

Correspondence between the Liouville equation and the Fokker-Planck equation:

Let us explain this result in the limit $D \to 0$ far from criticality. We warn the reader that we find something a little different from ref. [2] in the end! Precisely (4.35), (4.36) and also (4.37) has to be modified because of a different sign from (4.9) using (4.5). ref. [2] claims that (4.37) corresponds with the Liouville equation. This is not true since (4.37) is not in accordance with (3.12). Take e.g. $\mu < 0$ in (4.37): we get $\sigma_n \to -|\mu|(n+1)$. On the other hand (3.12) gives $s_n = -|\mu|n$.

Now to the theorem: We assume that our 1-d vector field F(x) only has a countable set of separated fixpoints $\{x_i\}$, where $F(x_i) = 0$ and $F'(x_i) = \lambda_i \neq 0$.

Then we construct the quantum-like potential locally around each fixpoint:

$$\Phi = \frac{1}{2}\nabla \cdot \vec{F} + \frac{1}{4D}F^2 \simeq +\frac{1}{2}\lambda_i + \frac{\lambda_i}{4D}(x - x_i)^2, \tag{69}$$

where we have a sign different from (4.35)! and used the approximation $F(x) \simeq \lambda_i(x - x_i)$. We remark that if (4.35) was to hold it would be inconsistent with (4.23) and (4.25). Anyway we then find the harmonic oscillator values to be

$$\epsilon_n = |\lambda_i|(n + 1/2) + \frac{\lambda_i}{2}. \tag{70}$$

Then changing to the Fokker-Planck equation gives us a sign of the above ϵ and we end up with

$$\sigma_{n,i} = -n|\lambda_i| - \frac{\lambda_i + |\lambda_i|}{2}. \tag{71}$$

Now ref. [2] says that in the limit $D \to 0$ the wells tend to be independent of Φ since the barriers grow. This is indeed true for the approximated Φ. Therefore the set of eigenvalues of the Fokker-Planck equation will gather all of the harmonic oscillators eigenvalues, $\{\sigma_{n,i}\}$, which were the Liouville eigenvalues.

9 Conclusion

We recapitulate: We have seen how to find the spectrum, the eigenfunctions and their duals for the dynamical system relevant partial differential equations: the Liouville and the Fokker-Planck equation. We have understood the interesting phenomena of continuous spectrum at criticality and a Jordan block decomposition of the former operators above criticality with respect to the pitchfork bifurcation. Furthermore we also now have a (at least physical) understanding of why noise destroys the continuos spectrum of the Liouville operator. Finally we have seen a correspondence theorem between the noise-less limit of the Fokker-Planck equation and the Liouville equation away from criticality.
I conclude this project by saying that I have found in the paper ref. [2] a very good introduction to spectral theory of the Liouville equation and the Fokker-Planck equation. The methods used give another, new view of the pitchfork bifurcation althougt the latter looks quite simple.

Perspectives:
Generalisation of the work on the spectral properties for the pitchfork bifurcation to other types of bifurcations, eg. Hopf etc. Nothing in the method of Gaspard et al. appears to depend crucially on that the bifurcation was the pitchfork. Of course the fact that this bifurcation was in 1-d simplified the matters: clever changes of variables, easy visualisation etc. But at least one could also in higher dimensions use numerical methods to check correspondence with the noise-less limit and the breakdown of the continuous spectrum when noise is added.

With respect to the dynamical averages it could be interesting to take a closer look on various cycle expansions ref. [1] of these with respect to the cases when the Liouville operator has a Jordan-block structure.

References

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