semiclassical!quantization
quantization!semiclassical
orbit!returning
returning orbit

Chapter 30

Semiclassical quantization

(G. Vattay, G. Tanner and P. Cvitanović)

We derive here the Gutzwiller trace formula and the semiclassical zeta function, the central results of the semiclassical quantization of classically chaotic systems. In chapter 32 we will rederive these formulas for the case of scattering in open systems. Quintessential wave mechanics effects such as creeping, diffraction and tunneling will be taken up in chapters 35 and 36. If you still feel you got no satisfaction, we derive the trace formulas one more time in appendix O, this time both for quantum and stochastic flows.

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30.1 Trace formula

Our next task is to evaluate the Green's function trace (26.16) in the semiclassical approximation. The trace

$$\mathrm{tr}\, G_{sc}(E) = \int d^D q\, G_{sc}(q,q,E) = \mathrm{tr}\, G_0(E) + \sum_j \int d^D q\, G_j(q,q,E)$$

receives contributions from "long" classical trajectories labelled by j which start and end in q after finite time, and the "zero length" trajectories whose lengths approach zero as $q' \to q$.

First we work out the contributions coming from the finite time *returning* classical orbits, that is, trajectories that originate and end at a given configuration point q. As we are identifying q with q', taking of a trace

add periodic trajectory to figure 30.1

Green's function!time dependent

30.5 Semiclassical zeta function

add a section on the $k = 0$ zeta function part

30.6 Noisy trace formula

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> In fact I'm Vattay Gábor, just these Indo-Europeans mix up the right order.
>
> (G. Vattay and P. Cvitanović)

The Fokker-Planck equation (29.14) can be viewed as the 'Schrödinger' problem of the 'quantized' Hamiltonian. The quantization means the introduction of operators $\hat{x} = x$ and $\hat{p} = -D\nabla$. D plays the role of $i\hbar$. The noisy commutators fulfill

$$[\hat{x}, \hat{p}] = D. \tag{30.20}$$

The Schrödinger equation is then

$$D\partial_t \psi = \hat{H}\psi = -D\nabla v\psi + D^2\nabla^2\psi. \tag{30.21}$$

The noise Hamilton operator is not a hermitian one, therefore the eigenenergies E_n defined as

$$\hat{H}\psi_n = E_n\psi_n \tag{30.22}$$

are complex numbers and their conjugates. Another special feature is that the eigenenergies of the Hamilton operator are proportional with D, due to the absence of a D independent potential term. It is then convenient to absorb the D dependence into $s_n = E_n/D$, which is then the eigenvalue of the Fokker-Planck equation.

not Feynman? this is Wiener path integral?

The Feynman path integral formalism can now be extended for this case. The time domain Green's function or in other words the evolution operator in Feynman formalism reads

$$\mathcal{L}^t(x, x') = \int \mathcal{D}x \exp\left(-\frac{1}{D}\int_0^t L(t')dt'\right), \tag{30.23}$$

where $\mathcal{D}x$ denotes the continuous limit of the usual discretized path summation for all possible paths connecting x with x' in time t and the integral in the exponent is the noise action computed for the path.

The analog of the Gutzwiller trace formula in this case is easy to write down:

$$
\begin{aligned}
Tr\mathcal{L}(E) &= \int_0^\infty dt e^{Et/D} \int dx \mathcal{L}^t(x,x') \\
&= \frac{1}{D} \sum_{p,r=1}^{\infty} \frac{e^{-\frac{r}{D}\int_0^{T_p} dt (L_p(t)-E) - ir\nu_p\pi}}{|\det(M_p^r - 1)|^{1/2}},
\end{aligned}
\tag{30.24}
$$

where the summation is over over all repaeats of the prime periodic orbits of Hamilton's equations. The action in the exponent, the monodromy matrix, and the topological index are the analogs of those in the Gutzwiller trace formula, with $i\hbar \to 2D$. The periodic orbits of the underlying noiseless dynamics are also the periodic orbits of the Hamilton's equations, with zero momentum $p = 0$. Other periodic orbits are possible. The monodromy matrix, action and topological index for the old periodic orbits have some nice features. [2] The monodromy matrix M_p then has two separate diagonal blocks, each of which is the Jacobian matrix $\mathbf{J}_p$ of the orbit in the noiseless dynamics. Using this, we can express the semiclassical amplitude as

link gutz. trace

how do you know this: "if the original dynamics is not uniformly hyperbolic"?

$$
\frac{1}{|\det(M_p^r - 1)|^{1/2}} = \frac{1}{|\det(\mathbf{J}_p^r - 1)|}.
\tag{30.25}
$$

The topological index counts the number of conjugate points. They come in pairs due to the structure of the matrix, so no phase factor goes into the exponent. For the deterministic periodic orbits, the Lagrangian is zero and by using $s = E/D$, the exponent becomes e^{srT_p}. Each deterministic periodic orbit p contributes to the trace formula a term of form

$$
\mathrm{tr}\, \mathcal{L}|_p \sim \frac{1}{D} \sum_r \frac{e^{srT_p}}{|\det(\mathbf{J}_p^r - 1)|},
$$

[3] This is exactly the classical trace formula.

eq link, CvitEck here

local minima, where variational searches fail

On the new periodic orbits, which were not present in the noiseless case, these simplifications cannot be carried out.

In the exponent, an extra $-\frac{1}{D}\int_0^{T_p} L_p(t)dt$ term shows up and the monodromy matrix cannot be factorized as above. The quantity $R_p = \int_0^{T_p} L_p(t)dt$ is a positive number independent of D. In the weak noise limit, these orbits are strongly suppressed due to the small term $e^{-R_p/D}$.

[2] Mason: define monodromy matrix, action and topological index

[3] Vattay: $1/D$ is just because the argument of the trace is E instead of s. In $Tr\mathcal{L}(s)$ the $1/D$ factor is not present.

conjunction
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We have shown the analogy of the quantum and noise problems. [4]

Commentary

⇓PRELIMINARY

Remark 30.1 Validity of the semiclassical quantization. For a scholarly discussion of the many puzzles that arise in the study of the validity and applicability of the semiclassical quantization developed above, see the review by Voros [?].

The main problem is that the semiclassical Van Vleck propagators are not multiplicative - the more appropriate term for the operation of joining them by evaluating saddlepoints is Mather's *conjunction* that plays the same role in the variational theory of area preserving twist maps.

Remark 30.2 Where does teh symplectic structure come from? The Hamiltonian or symplectic structure of the equations describing the most probable path was not present in the original deterministic equation $\dot{x} = v(x)$. It is a property of the approximation we made. Instead of x and p, one can introduce other canonical coordinates via a canonical transformation. But then their 'quantization' will not reproduce the Fokker-Planck equation. We get the Fokker-Planck equation only if the system is quantized in Cartesian coordinates. We can say that the symplectic symmetry is present only on the first level of the approximation. Thinking about the correspondence between classical and quantum mechanics, we see that the Hamilton equations of motion are the first approximation of quantum mechanics. However, there is no way to quantize systems in general canonical coordinates. In this line of thought, the symplectic structure is not a deep symmetry of nature, but an artifact of a semiclassical approximation to quantum dynamics. (Gábor Vattay)

⇑PRELIMINARY

Remark 30.3 Zeta functions. For "zeta function" nomenclature, see remark 13.4 on page 287.

Résumé

to "Why does it work"?

Spectral determinants and dynamical zeta functions arise both in classical and quantum mechanics because in both the dynamical evolution can be

[4] Mason: you have not