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1 Extraction of unstable periodic orbits from noisy data

There are two main methods used to detect unstable periodic orbits in experimental data. The first, and conceptually simpler, one was developed by Moss (Pierson and Moss, 1995). For fixed points, the basic algorithm proceeds as follows. The delay-coordinate embedded data is examined for a series of points that approaches a putative fixed point within a certain radius. The algorithm looks for three points that approach the fixed point and three points that recede from it, in an approximately linear fashion. Two least-squares lines are fit to the approaching and receding points, and these are considered the stable and unstable manifolds, respectively. The intersection of the two lines is considered to be the fixed point, as long as it is within a certain distance from the identity ($y = x$) line.

The second method was developed by So (So *et al.* 1997; So *et al.* 1996) and uses a mathematical transform to concentrate the data around unstable periodic orbits so that they show up as sharp peaks in a histogram (i.e., a distribution function) of the data. This is the technique used in this project.

1.1 Unstable periodic orbit detection

The method derived by So *et al.* (1997) is used to detect unstable periodic orbits in each experiment. This technique uses a transformation to concentrate data around unstable periodic orbits, making it easy to identify them in a histogram of the data.

1.2 Fixed point transform

For fixed points, the transformation is defined as

$$s1image0.gif \tag{1}$$

where z is the d-dimensional delay-coordinate vector of IBIs, I is the identity matrix, $F(z) = z_{i+1}$, and $S(z, R)$, a $d \times d$ matrix function of z and a $d \times d \times d$ random tensor R, is given by

$$S(z, R) = gradF(z) + R \cdot [F(z) - z] \tag{2}$$

Here $F(z)$ is the $d \times d$ Jacobian matrix of $F(z)$, which is calculated using a least-squares fit of 3 spatial nearest neighbors (i.e., the nearest 3 points in phase space). This transform is applied to every point z in the set and summarized in a one-dimensional distribution function of the experimental data approximated by a histogram with a bin size of 0.02 seconds. The transformation shifts all points in the linear region of a fixed point $z*$ even closer to $z*$, thus creating a peak in the distribution function $\rho(\hat{z})$. At any one value of R, the transformation produces spurious peaks; these are filtered out by averaging $\rho(\hat{z})$ over several (300) different values of R. The higher the magnitude (ρ) of R, the more the peaks are smoothed out and the lower their amplitude.

The significance of these peaks is assessed using surrogate data controls. Fifty phase-randomized surrogates are generated and transformed, and their corresponding distributions $\rho_{surr}((\hat{z})$ are calculated and averaged together. The deviation from the surrogate mean $\rho_{surr}(\hat{z})$ for a given surrogate is denoted by $w(\hat{z}) = \rho_{surr}(\hat{z}) - \rho_{surr}(\hat{z})$ and the maximum deviation of each surrogate over all $\hat{z}$ is defined as $W = 3Dmax(w(\hat{z})))$. Then the fraction of surrogates (W) with deviations W exceeding W is calculated. When enough surrogates are used, this fraction approximates the probability that W exceeds W. Using the distribution of (W) vs. W, the maximum deviations of the experimental data distributions $(\rho(\hat{z}) - \rho_{surr}(\hat{z}))$ are compared against the surrogates, providing the probability (or P value) that these peaks are due to the null hypothesis. The peaks are considered significant for a P value of 0.04 or less; this is considered evidence of a detected unstable periodic orbit.

One problem with biological time series data is the system's *nonstationarity*. That is, the location of a periodic orbit can drift over time. This drift can be due to variations in external parameters, such as bath temperature or flow rate, or intrinsic biological properties. In any case, this phenomenon causes peaks to be spread out in So's transform approach. To counteract this effect, each data set is broken up into short windows of time; this minimizes the drift. However, the shorter the window, the smaller the peaks will be, since the likelihood of encountering a unstable periodic orbit will decrease. Therefore, the choice of window size is a careful tradeoff between signal amplitude and noise reduction.

1.3 Periodic orbit transform for higher periods

For unstable periodic orbits of periods greater than one, the transform becomes slightly more complicated. For a period p periodic orbit, one could use the above (fixed point) transform by making delay vectors using delay $p = B7$ instead of . However, this approach would require that both $z(n)$ and $z(n+p)$ be very close to $z*$. It is unlikely to find a set of such points when the system is noisy or has a large maximal Lyapunov exponent, especially as p gets large. Therefore, So *et al.* (1997) propose a method that uses every iterate instead of only every pth iterate. Briefly, this method proceeds as follows. Assume a given periodic orbit of period p, $z*(1), ..., z*(p)$, such that $F^p(z*(1)) = z*(p) = z*(1)$. There are p points along this periodic orbit which is labeled $z*(k)$, where $k = 1, ..., p$. At each point $z*(k)$ along the orbit the local dynamics can be approximated , as in the fixed point transform, with a first-order Taylor series approximation:

$$F(z)z*(k+1) + F(z) = [z - z*(k)] , \tag{3}$$

where $F(z)$, the Jacobian matrix, is calculated using a least squares fit. Then one takes a set of p points z_k, $k = 1, ..., p$, chosen from the delay-embedded data set $z(n)$ and applies the same approximation at each point. The "periodic orbit" transform is then defined as

$$s1image2.gif \tag{4}$$

where $Z = (z_1, ..., z_p)$ is a dp-dimensional column vector that is mapped to $\hat{z}$ by the transform G, I is the $d \times d$ identity matrix, and $R = R_1, ..., R_p$ is a set of p random $d \times d \times d$ tensors which serve the same purpose as in the fixed point transform. The $d \times d$ matrix S is defined as

$$S(z, z', R) = F + R[F(z) - z'] . \tag{5}$$

As in the fixed point transform, G shifts orbits that are near the periodic orbit even closer to the unstable periodic orbits, and spreads the other orbits out.

This transform is applied to groups of p data points and their closest *spatial* neighbors. This facilitates the use of all components of an orbit, because even if only some of the p components are close to a periodic orbit, other sequences formed from their neighbors may stay close to the periodic orbit for the entire cycle and thus be mapped closer to the unstable periodic orbit. The resulting transformed data is distilled into a distribution function, ideally p-dimensional, but usually 1- or 2-dimensional for clarity. This distribution function $\rho(\hat{z})$ is again approximated by a histogram with bin size 0.02 s. Peaks in this histogram will thus denote the locations of all possible p components of period-p orbits. These peaks can again be compared with surrogate data to obtain an estimate of their significance.

References

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