Term project

Spatiotemporally periodic solutions by variational methods

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1 Hopf's last hope, or: Turbulence, and what to do about it?

Hopf[1] and Spiegel[2, 3, 4] have proposed that the turbulence in spatially extended systems be described in terms of recurrent spatiotemporal patterns. Pictorially, dynamics drives a given spatially extended system through a repertoire of unstable patterns; as we watch a turbulent system evolve, every so often we catch a glimpse of a familiar pattern. For any finite spatial resolution, for a finite time the system follows approximately a pattern belonging to a finite alphabet of admissible patterns, and the long term dynamics can be thought of as a walk through the space of such patterns, just as chaotic dynamics with a low dimensional attractor can be thought of as a succession of nearly periodic (but unstable) motions.

Hopf's proposal is in its spirit very different from most ideas that animate current turbulence research. It is distinct from the Landau quasiperiodic picture of turbulence as a sum of infinite number of incommensurate frequencies, with dynamics taking place on a large-dimensional torus. It is not the Kolmogorov's 1941 homogeneous turbulence with no coherent structures fixing the length scale, here all the action is in specific coherent structures. And it is *not* probabilistic; everything is fixed by the deterministic dynamics with no probabilistic assumptions on the velocity distributions or external stochastic forcing.

Carrying out Hopf's program in a systematic manner is an open research problem towards which we hope to contribute in this project. More gener-

ally, we have to move to PDEs to go forward with the study of dynamical systems, and one of the main problems facing us is how to find periodic solutions for such systems.

Our goal here is to develop a new method or methods for finding spatiotemporally periodic solutions of equations describing extended systems. The idea is to make an informed rough guess of what the solution looks like, and then use variational methods to drive our guess toward the exact solution. If the periodic orbits are isolated, as they are in chaotic systems, this approach should be very robust.

The project is laid out in two parts; the first part reviews a particular example already worked out in reference [9]. Reproducing these results is useful for developing intuition about such problem, and confidence in correctness of your numerical methods, but is not an essential prerequisite to the second part.

The real work starts with the second part, sect. 8; develop new methods for determining spatiotemporally periodic solutions. If you are the impatient sort, start there.

1.1 The warm-up

The only way to developed intuition about chaotic dynamics is by computing, and you are urged to try to work through the essential steps in this project, combining the techniques learned in courses on nonlinear dynamics with whatever other explorations that seem of of interest to you. The steps are:

1. a problem is posed and formulated - here as the equation (1) describing the dynamics of a flame front.
2. ways of reformulating the dynamics suited to numerical explorations are explored. Here the Fourier representation (5) seems a good starting point.
3. determine the fixed point(s)
4. compute the fixed point stabilities

5. explore the fixed point stabilities and the associated bifurcations as system parameter(s) are varied

So far all work has been analytic, and you have developed some feeling for the topology of solutions and bifurcation sequences. This is all one can extract from the problem by analysis not assisted by numerical experimentation. Next,

1. implement a numerical simulator for your problem
2. plot a variety of long orbits to get some sense for the attractors for different values of system parameters
3. find numerically stable cycles, if any
4. compute stable cycle stabilities
5. determine values of parameters for stable cycles $\rightarrow$ unstable cycles bifurcations (at least estimate by trial and error)
6. diagnose parameter values at which onset of chaos seems to be taking place
7. firm up your hunch by estimating by numerical simulation some chaos diagnostic, like existence of a positive Lyapunov exponent
8. try to study such bifurcations in a Poincaré section
9. compute the unstable cycle eigenvectors
10. if a cycle is unstable, and you have succeeded in computing its unstable eigenvector(s), attempt to trace out its unstable manifold by starting with a set of points close to its Poincaré section fixed point, sprinkled along the unstable eigenvector
11. have a Carlsberg, perhaps the best beer in parts of Copenhagen where they serve no other kind

In the project tour guide that follows, we have indicated which steps are project questions, and which are optional. We do not expect you to get through all of the points discussed; do as much as you feel needs doing.

The list of references is appended only to amuse you, and none are needed in order to get through the project.

2 Fluttering flame front

We explore such ideas in a spatially extended system claimed to describe the flutter of the flame front of gas burning in a cylindrically symmetric burner on your kitchen stove. In a few days of analysis you should by able to simulate the system numerically, and develop some intuition about such systems on the level of nonlinear dynamics textbooks: determine the fixed points, stabilities, stability eigenvectors, bifurcations, onset of chaos.

The Kuramoto-Sivashinsky equation[5, 6] is one of the simplest partial differential equations that exhibits chaos. It is a dynamical system extended in one spatial dimension, defined by

$$u_t = (u^2)_x - u_{xx} - \nu u_{xxxx} \,. \tag{1}$$

In this equation $t \geq 0$ is the time and $x \in [0, 2\pi]$ is the space coordinate. The subscripts x and t denote the partial derivatives with respect to x and t; $u_t = du/dt$, u_{xxxx} stands for 4th spatial derivative of the "height of the flame front" $u = u(x,t)$ at position x and time t. ν is a "viscosity" damping parameter; its role is to suppress solutions with fast spatial variations. The term $(u^2)_x$ makes this a *nonlinear system*.

How good description of a flame front this is need not concern us here; suffice it to say that such model amplitude equations for interfacial instabilities arise in a variety of contexts - see e.g. reference [7] - and this one is perhaps the simplest physically interesting spatially extended nonlinear system. The salient feature of such partial differential equations is that for any finite value of the phase-space contraction parameter ν a theorem says that the asymptotic dynamics is describable by a *finite* set of "inertial manifold" ordinary differential equations[8]. We shall verify by numerical experimentation that even though $u(x,t)$ is in principle infinite dimensional (it has a component for each spatial point x), the attractors are indeed of finite dimension.

The "flame front" $u(x,t) = u(x + 2\pi, t)$ is periodic on the $x \in [0, 2\pi]$ interval, so the standard strategy is to expand it in a discrete spatial Fourier series:

$$u(x,t) = \sum_{k=-\infty}^{+\infty} b_k(t) e^{ikx} \,. \tag{2}$$

Since $u(x,t)$ is real,

$$b_k = b^*_{-k} \,. \tag{3}$$

Show that substituting (2) into (1) yields the infinite ladder of evolution equations for the Fourier coefficients b_k: Project question

$$\dot{b}_k = (k^2 - \nu k^4) b_k + ik \sum_{m=-\infty}^{\infty} b_m b_{k-m} \,. \tag{4}$$

As $\dot{b}_0 = 0$, the solution integrated over space is constant in time. In what follows we shall assume that this average is zero, $\int dx\, u(x,t) = 0$.

The coefficients b_k are in general complex functions of time. We can simplify the system (4) further by considering the case of b_k pure imaginary, $b_k = i a_k$, where a_k are real, with the evolution equations

$$\dot{a}_k = (k^2 - \nu k^4) a_k - k \sum_{m=-\infty}^{\infty} a_m a_{k-m} \,. \tag{5}$$

Argue that this picks out the subspace of odd solutions $u(x,t) = -u(-x,t)$ (the optional reading in sect. 6 discusses this in more detail). Use (3) to further simplify the tower of evolution equations. Project question

How you solve the equations numerically is up to you. Here are some of the options:

1. You can divide the x interval into a sufficiently fine discrete grid of N points, replace space derivatives (1) by approximate discrete derivatives, and integrate a finite set of first order differential equations for the discretized spatial components $u_j(t) = u(2\pi j/N, t)$, by any integration routine you feel comfortable with (a variant of this is discussed in sect. 8.1).

2. You can integrate numerically the Fourier modes (5), truncating the ladder of equations to a finite length N, i.e., set $a_k = 0$ for $k > N$. In our experience, for this exploration $N \leq 16$ truncations were sufficiently accurate.

3. If you happen to have such things handy, you can use more sophisticated numerical methods, such as pseudo-spectral methods or implicit methods.

If your integration takes days and gobbles up terabits of memory, you are using brain-damaged "high level" software. You should have written a few lines of the Runge-Kuta code in some mundane everyday language.

In our own simulations, we have have determined the solutions in the space of Fourier coefficients, and then reconstituted from them the spatiotemporal solutions of (1).

The trivial solution $u(x,t) = 0$ is a fixed point of (1).

Show that from (5) it follows that the $|k| < 1/\sqrt{\nu}$ long wavelength modes of this fixed point are linearly unstable, and the $|k| > 1/\sqrt{\nu}$ short wavelength modes are stable. For $\nu > 1$, $u(x,t) = 0$ is the globally attractive stable fixed point, i.e., the dissipation is so strong that any flame front burns out. Project question

Starting with $\nu = 1$ the solutions go through a rich sequence of bifurcations, studied e.g. in reference [7].

What kind of bifurcation takes place as $\nu > 1 \to \nu < 1$? As ν decreases, are there any further bifurcations from the $u(x,t) = 0$ fixed point, and if so, of what type? Project question

Try to determine some further fixed points of (5); if any, discuss bifurcations that lead to them, etc. (We have not checked myself in any significant detail what interesting fixed points are there beyond $u(x,t) = 0$, but there is surely a whole zoo, so do not spend too much time on this.) Optional

Detailed investigation of the parameter dependence of bifurcations sequences is in general so laborious that it might totally derail you from the path to the main goals of this project. Wisest to turn to numerical experimentation. We shall take $\sqrt{\nu}$ sufficiently small so that the dynamics can be spatiotemporally chaotic, but not so small that we would be overwhelmed by too many short wavelength modes needed in order to accurately represent the dynamics.

Some advice on how to do such exploration: start on *terra firma*, at $\nu = 1$, low N, and decrease ν a little bit, integrate until the trajectory has settled

down; then decrease ν a little bit again, integrate until the trajectory has settled down. Repeat. Stop incrementing ν for a bit, increment N instead and check how sensitive is your attractor to truncation size. You will sail thorough a sequence of bifurcations and enter chaos, most likely via period-doubling route. This "adiabatic" approach has advantage of (almost) always starting you close to the attractor and avoids potentially long transients of arbitrary starting conditions.

3 Fourier modes truncations

The growth of the unstable long wavelengths (low $|k|$) excites the short wavelengths through the nonlinear term in (5). The excitations thus transferred are dissipated by the strongly damped short wavelengths, and a sort of "chaotic equilibrium" can emerge. The very short wavelengths $|k| \gg 1/\sqrt{\nu}$ will remain small for all times, but the intermediate wavelengths of order $|k| \sim 1/\sqrt{\nu}$ will play an important role in maintaining the dynamical equilibrium. Hence, while one may truncate the high modes in the expansion (5), care has to be exercised to ensure that no modes essential to the dynamics are chopped away. In practice one does this by repeating the same calculation at different truncation cutoffs N, and making sure that inclusion of additional modes has no effect within the accuracy desired.

When we simulate the equation (5) on a computer, we have to truncate the ladder of equations to a finite length N, i.e., set $a_k = 0$ for $k > N$. N has to be sufficiently large that no harmonics a_k important for the dynamics with $k > N$ are truncated. On the other hand, computation time increases with the increase of N. For reasons that will be explained below, we have performed our numerical calculations taking $N = 16$.

The problem with such high dimensional truncations of (5) is that the dynamics is difficult to visualize. Best we can do without much programming is to examine trajectory's projections onto any three axes a_i, a_j, a_k.

Plot your trajectory for the same ν, the same two or three axes as in figure 1; is your dynamics qualitatively the same as in our plots? Project question

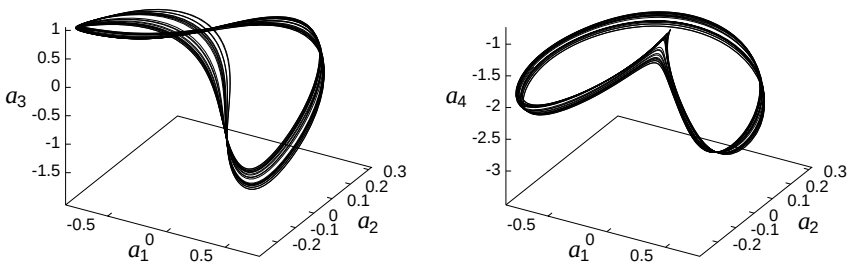

Figure 1: Projections of a typical 16-dimensional trajectory onto different 3-dimensional subspaces, coordinates (a) $\{a_1, a_2, a_3\}$, (b) $\{a_1, a_2, a_4\}$. $N = 16$ Fourier modes truncation with $\nu = 0.029910$.

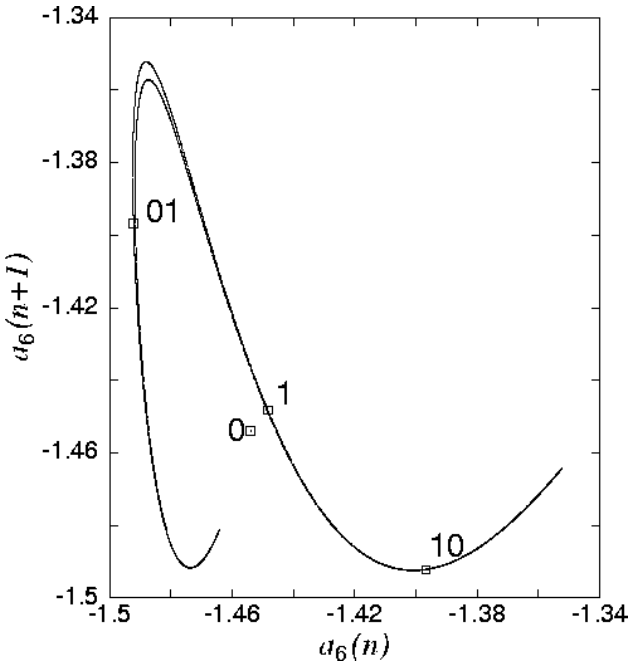

Figure 2: The attractor of the system (5), plotted as the a_6 component of the $a_1 = 0$ Poincaré section return map, 10,000 Poincaré section returns of a typical trajectory. Indicated are the periodic points $\overline{0}$, $\overline{1}$ and $\overline{01}$. $N = 16$ Fourier modes truncation with $\nu = 0.029910$.

4 Poincaré section

Optional

The question is how to look at such flow? Usually one of the first steps in analysis of such flows is to restrict the dynamics to a Poincaré section. We fix (arbitrarily) the Poincaré section to be the hyperplane $a_1 = 0$, and integrate (5) with the initial conditions $a_1 = 0$, and arbitrary values of the coordinates $a_2, \ldots, a_N$, where N is the truncation order. When a_1 becomes 0 the next time, the coordinates $a_2, \ldots, a_N$ are mapped into $(a_2', \ldots a_N') = P(a_2, \ldots, a_N)$, where P is the Poincaré mapping of a $N-1$ dimensional hyperplane into itself. Figure 2 is an example of a results that one gets. While the topology of the attractor is still obscure, one thing is clear - as claimed in the introduction, the attractor is finite and thin, barely thicker than a line.

5 Bifurcation trees

Optional

Provided we have figured out how to generate numerically a Poincaré section, we can let computer run and do some numerical fishing on our behalf. Figure 3 is a representative bifurcation diagram for the system at hand. To obtain this figure, we took a random initial point, iterated it for a some time to let it settle on the attractor and then plotted the a_6 coordinate of the next 1000 intersections with the Poincaré section. Repeating this for different values of the damping parameter ν, one can obtain a picture of the attractor as a function of ν. For an intermediate range of values of ν, the dynamics exhibits a rich variety of behaviors, such as period-doubling, strange attractors, stable limit cycles, and so on.

We have found that the minimum value of N to get any chaotic behavior at all was $N = 9$.

Do you get any chaos for $N < 9$?

Project question

The dynamics for the $N = 9$ truncated system is rather different from the full system dynamics, and therefore we have performed all calculations reported here for $N = 16$, which seemed a reasonable cutoff. Having been there, done that, we recommend examining in particular two values of the damping parameter: $\nu = 0.029910$, for which the system is chaotic, and $\nu = 0.029924$, for which the system has a stable period-3 cycle.

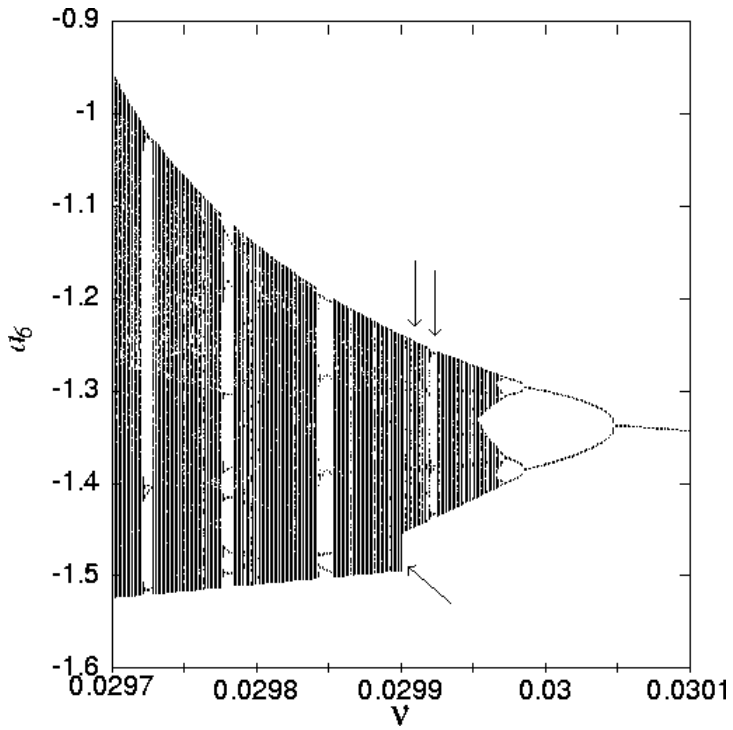

Figure 3: Period-doubling tree for coordinate a_6, $N = 16$ Fourier modes truncation of (5). The two upper arrows indicate the values of damping parameter that we use in our numerical investigations; $\nu = 0.029910$ (chaotic) and $\nu = 0.029924$ (period-3 window). Truncation to $N = 17$ modes yields a similar figure, with values for specific bifurcation points shifted by $\sim 10^{-5}$ with respect to the $N = 16$ values. The choice of the coordinate a_6 is arbitrary; projected down to any coordinate, the tree is qualitatively the same.

Do you get a stable cycle for $\nu = 0.029924$? Project question

6 Symmetry decomposition

Optional

Before proceeding with the calculations, we take into account the symmetries of the solutions. Consider the spatial flip and shift symmetry operations $Ru(x) = u(-x)$, $Su(x) = u(x + \pi)$. The latter symmetry reflects the invariance under the shift $u(x,t) \to u(x + \pi, t)$, and is a particular case of the translational invariance of the Kuramoto-Sivashinsky equation (1). In the Fourier modes decomposition (5) this symmetry acts as $S : a_{2k} \to a_{2k}, a_{2k+1} \to -a_{2k+1}$. Relations $R^2 = S^2 = 1$ induce decomposition of the space of solutions into 4 invariant subspaces[7]; the restriction to $b_k = ia_k$ that lead to simplified set of equations (5) amounts to specializing to a subspace of odd solutions $u(x,t) = -u(-x,t)$.

Now, with the help of the symmetry S the whole attractor $\mathcal{A}_{tot}$ can be decomposed into two pieces: $\mathcal{A}_{tot} = \mathcal{A}_0 \cup S\mathcal{A}_0$ for some set $\mathcal{A}_0$. It can happen that the set $\mathcal{A}_0$ (the symmetrically decomposed attractor) can be decomposed even further into four disjoint sets: $\mathcal{A}_{tot} = \mathcal{A} \cup S\mathcal{A} \cup \Theta\mathcal{A} \cup \Theta S\mathcal{A}$.

7 Tour of a few numerical results

If we are integrating an unstable, chaotic solution in the Fourier space, we can go back to the configuration space using (2) and plot the corresponding spatiotemporal solution $u(x,t)$.

Plot a spatiotemporal solution $u(x,t)$ for the chaotic, $\nu = 0.029910$ attractor. Project question

Staring at the solution as it evolves in time we should start getting a glimpse of the repertoire of the spatiotemporal patterns that Hopf wanted us to see in turbulent dynamics.

Hopf's proposal for a theory of turbulence was to think of turbulence as a sequence of near recurrences of a repertoire of unstable spatiotemporal patterns. The project so far falls short of implementing the proposal, but it

sheds some light on how such ideas are developed - numerical solutions that you have studied are both "turbulent" and recognisable to the eye.

8 Spatiotemporally periodic solutions by variational methods

Hopf wanted us to see *recurrent* patterns, that is to say, the unstable spatiotemporally periodic solutions of our equations. This can be done, but is hard work - we list a few computed by Freddy Christiansen in table 1, and plot the shortest one in figure 4, just to give you a feeling for the form and stability of such solutions. Other solutions exhibit the same overall gross structure - a few wiggles here and there, continuously in flux and yet so alike.

Finding a periodic solution amounts to solving an equation, and root finding methods are the first choice. But in higher dimensions, we may have to settle for almost periodic orbits, as getting them right may turn out too expensive.

If we concentrate on diffusion-like equations (such as the Kuramoto-Sivashinsky), then one is going to have a finite dimensional attractor (the inertial manifold) and there will only be a small number of expanding directions. Root finding techniques only need to control the expanding directions, as the contracting ones will converge just by time iteration. It is possible to rig Newton's method so that one only has to invert a matrix containing the expanding directions. In this direction, Yueheng Lan has already implemented the singular-value-decomposition inversion of the stability matrix needed for the Newton-Raphson method.

One of the objectives of a theory of turbulence is to predict measurable global averages over turbulent flows, such as velocity-velocity correlations and transport coefficients. With the present parameter values we are far from any strongly turbulent regime, and in fact we are happy if we manage to implement even the simplest test of chaotic dynamics: evaluation of the Lyapunov exponents.

Project question

For the strange attractor at $\nu = 0.029910$ our numerical simulation estimate for the **Lyapunov exponent** is 0.629. What do you get?

Table 1: A few unstable cycles for the $N = 16$ Fourier modes truncation of the Kuramoto-Sivashinsky equation (5), damping parameter $\nu = 0.029910$ (chaotic attractor) and $\nu = 0.029924$ (period-3 window), periods, the first four stability eigenvalues. The deviation from unity of Λ_2, the eigenvalue along the flow, is an indication of the accuracy of the numerical integration.

p	T_p	Λ_1	$\Lambda_2 - 1$	Λ_3	Λ_4
Chaotic, $\nu = 0.029910$					
0	0.897653	3.298183	$5{\cdot}10^{-12}$	$2.793085{\cdot}10^{-3}$	$2.793085{\cdot}10^{-3}$
1	0.870729	-2.014326	$5{\cdot}10^{-12}$	$6.579608{\cdot}10^{-3}$	$3.653655{\cdot}10^{-4}$
10	1.751810	-3.801854	$8{\cdot}10^{-12}$	$3.892045{\cdot}10^{-5}$	$2.576621{\cdot}10^{-7}$
Period-3 window, $\nu = 0.029924$					
0	0.897809	3.185997	$7{\cdot}10^{-13}$	$2.772435{\cdot}10^{-3}$	$-2.772435{\cdot}10^{-3}$
1	0.871737	-1.914257	$5{\cdot}10^{-13}$	$6.913449{\cdot}10^{-3}$	$-3.676167{\cdot}10^{-4}$
10	1.752821	-3.250080	$1{\cdot}10^{-12}$	$4.563478{\cdot}10^{-5}$	$2.468647{\cdot}10^{-7}$

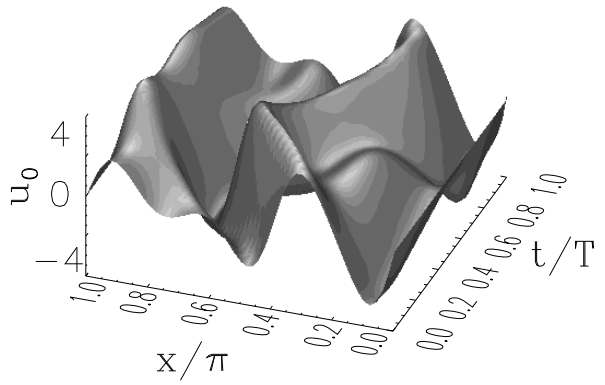

Figure 4: Spatiotemporally periodic solution $u_0(x,t)$. We have divided x by π and plotted only the $x > 0$ part, since we work in the subspace of the odd solutions, $u(x,t) = -u(-x,t)$. $N = 16$ Fourier modes truncation with $\nu = 0.029910$.

Consider again the Kuramoto-Sivashinsky system (1). By assumption, $u(x,t) = u(x+2\pi,t)$ is periodic on the $x \in [0,2\pi]$ interval. Assume that you have found a periodic solution $u(x,t) = u(x,t+T)$ with time period T. If you rescale the time $t \to T$ (and forget about an overall 2π factor), $u(x,t)$ is doubly periodic on the torus $u(x,t) = u(x+2\pi,t) = u(x,t+2\pi)$. So our goal is to determine solutions on the elementary cell $x \in [-\pi,\pi]$, $t \in [-\pi,\pi]$ with periodic boundary conditions both in spatial and time directions.

8.1 Discrete grid version

Divide the elementary cell into M time intervals and N space intervals, so $u(x,t)$ is replaced by the array u_{ij}. In order to study the discretization of the equation (1), define the $[M \times N]$ array $F(u)_{ij}$ defined by discretization of

$$F(u) = -u_t + (u^2)_x - u_{xx} - \nu u_{xxxx}\,. \tag{6}$$

So our task is to find values of u_{ij} satisfying the doubly-periodic boundary conditions such that such that $F(u)_{ij} = 0$; any such solution is a spatiotemporally periodic solution.

In order to implement this we need to define *lattice derivatives*. What follows is cribbed from reference [10] and perhaps not too relevant - you might want to skip this and go directly to sect. 8.4 for now. Axel Brandenburg (`www.nordita.dk/~brandenb/`) promises to give us more sophisticated derivatives, defined over more points.

8.2 Lattice derivatives

Consider a smooth function $u(x)$ evaluated on d-dimensional lattice

$$u_\ell = u(\ell)\,, \qquad\qquad \ell = \text{lattice point}\,,\ N^d \text{ points in all}\,.$$

A vector u specifies a lattice configuration. Assume the lattice is (hyper)cubic, and let $\hat{n}_\mu \in \{\hat{n}_1, \hat{n}_2, \cdots, \hat{n}_d\}$ be the lattice cell vectors pointing

along the d positive directions, $|\hat{n}_\mu| = a$, where a is the lattice spacing. The *lattice derivative* is then

$$(\partial_\mu u)_\ell = \frac{u_{\ell+\hat{n}_\mu} - u_\ell}{a} .$$

Anything else with the correct $a \to 0$ limit would do, but this is the simplest choice. We can rewrite the derivative as a linear operator, by introducing the *hopping operator* (or "shift", or "step") in the direction μ

$$(\mathbf{h}_\mu)_{\ell j} = \delta_{\ell+\hat{n}_\mu, j} . \tag{7}$$

We write $\mathbf{h}$ out for the 1-dimensional case in its full $[N \times N]$ matrix glory:

$$\mathbf{h} = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & & \ddots & \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{pmatrix} . \tag{8}$$

We will assume throughout that the lattice is *periodic* in each $\hat{n}_\mu$ direction.

Applied on the lattice configuration $u = (u_1, u_2, \cdots, u_N)$, the hopping operator shifts the lattice by one site, $\mathbf{h}u = (u_2, u_3, \cdots, u_N, u_1)$. Its transpose shifts the entries the other way, so the transpose is also the inverse

$$\mathbf{h}^{-1} = \mathbf{h}^T . \tag{9}$$

The lattice derivative can now be written as a multiplication by a matrix:

$$\partial_\mu u_\ell = \frac{1}{a} (\mathbf{h}_\mu - \mathbf{1})_{\ell j} u_j .$$

In the 1-dimensional case the $[N\times N]$ matrix representation of the lattice derivative is:

$$\partial = \frac{1}{a}\begin{pmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ 1 & & & & & -1 \end{pmatrix} . \tag{10}$$

To belabor the obvious: On a finite lattice of N points a derivative is simply a finite $[N\times N]$ matrix. Continuum field theory is a world in which the lattice is so fine that it looks smooth to us. Whenever someone utters the word "operator" you say "it is just a matrix, no big deal, I know what a matrix is". OK?

8.3 Lattice Laplacian

In order to get rid of some of the lattice indices in "actions" it is convenient to employ vector notation for the terms bilinear in u, and keep the rest lumped into an "interaction" action,

$$S[u] = -\frac{\beta r'}{2} u^T \cdot u - \frac{\beta c}{2a^2} \left[(\mathbf{h}_\mu - \mathbf{1})\, u \right]^T \cdot (\mathbf{h}_\mu - \mathbf{1})\, u + S_I[u] \,. \tag{11}$$

In the continuum integration by parts moves ∂_μ around; on a lattice this amounts to a matrix transposition

$$\left[(\mathbf{h}_\mu - \mathbf{1})\, u \right]^T \cdot \left[(\mathbf{h}_\mu - \mathbf{1})\, u \right] = u^T \cdot (\mathbf{h}_\mu^{-1} - \mathbf{1})\, (\mathbf{h}_\mu - \mathbf{1}) \cdot u \,.$$

If you are wandering where the "integration by parts" minus sign is, it is there in discrete case at well. It comes from the identity $\partial^T = -\mathbf{h}^{-1}\partial$. The combination $\Delta = \mathbf{h}^{-1}\partial^2$

$$\Delta = -\frac{1}{a^2} \sum_{\mu=1}^{d} (\mathbf{h}_\mu^{-1} - \mathbf{1})\, (\mathbf{h}_\mu - \mathbf{1}) = -\frac{2}{a^2} \sum_{\mu=1}^{d} \left(\mathbf{1} - \frac{1}{2}(\mathbf{h}_\mu^{-1} + \mathbf{h}_\mu) \right) \tag{12}$$

is the *lattice Laplacian*. It is the simplest spatial derivative allowed for $x \to -x$ symmetric actions. In the 1-dimensional case the $[N \times N]$ matrix representation of the lattice Laplacian is:

$$\Delta = \frac{1}{a^2} \begin{pmatrix} -2 & 1 & & & & 1 \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & 1 & & \ddots & \\ & & & & & 1 \\ 1 & & & & 1 & -2 \end{pmatrix} . \tag{13}$$

The lattice Laplacian measures the second variation of a field u_ℓ across three neighboring sites. You can easily check that it does what the second derivative is supposed to do by applying it to a parabola restricted to the lattice, $u_\ell = u(\ell)$, where $u(\ell)$ is defined by the value of the continuum function $u(x) = x^2$ at the lattice point ℓ.

8.4 Search for periodic solutions

In the above we obtained a space-time periodic solution by assuming that we know the time period T; space periodicity is fixed by the construction. Of course, we do not know the time period *a priori*, so in seeking a variational solution of $F(u, T) = 0$ we should vary MN components of u_{ij} together with the time scale T.

The idea of discretizing in space and time and then minimizing an error may work, but it also might generate too many spurious solutions. These would be high frequency solutions that have nothing to do with the dynamics.

Discretized PDE condition $F(u) = 0$ can be solved by a method like Newton-Raphson, provided derivative matrix and its inverse have efficient representations. This might be possible if one keeps terms to leading order in lattice spacing.

Another alternative is to define a cost function

$$E(u) = \sum_{ij} F(u)_{ij} F(u)_{ij} \tag{14}$$

(or something like this, positive definite, but with a choice of metric more informed than just the straight Euclidean one), and then use a multidimensional conjugate gradient method [11]. We am not sure why looking for zeroes of $F(u)^2$ should be faster/better/more robust than looking for zeroes of $F(u)$, but time will tell. Brandenburg, Jayaprakash think so.

Discretized PDE solution $F(u) = 0$ is invariant separately under $u \to u + \text{const.}$, $x \to x + \text{const.}$, $t \to t + \text{const.}$, so these invariances have to be broken for numerics to diverge.

8.5 Fourier space version

Here the idea is to require that a time periodic solution of (5) exists for T to be determined, Fourier expand $a_k(t) \to a_{k\omega}$ in this period, truncate also in the number of time components $\omega < M$, solve the nonlinear equations in the NM components $a_{k\omega}$ so obtained, and hope it all converges well.

This is possibly known as the Carleman embedding - we are not sure. Refs. [15]-[19] might be relevant.

The Fourier inversion method is possibly not too bad. Mainieri has tried to reproduce Helleman results [12, 13], by feeding the resulting algebraic equations to Mathematica to solve. In a system with no pruning, all possible root are are real; in a system with pruning, some roots become complex, so the motivation was to use algebra results to figure out the pruning. (References [12, 13] are not easy to track down; if needed, Predrag has found them through the Niels Bohr Institute Library.)

The problem is really one of intersection of curves, algebraic geometry, a field which has seen a plethora of results in recent times. Numerically, one could use root tracking methods, and one may be able to find 1000 or so real roots. One may also have a handle on pruning by assuming that all roots are real at first and then trying to eliminate families of roots.

8.6 A warmup: temporally periodic solutions by variational methods

Starting with PDE discretization sect. 8.1 is probably unwise - the simplest thing to test first is how a *time discretized* ODE performs. So one could take only 3 spatial "lattice points", for example the *Rössler flow*

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ayz \\ \dot{z} &= b + z(x - c)\,, \qquad a = b = 1/5\,, \quad c = 5.7 \end{aligned} \tag{15}$$

and discretized (or Fourier transform) the time. We know that for these parameter values the system is chaotic and has lots of periodic solutions.

9 What next?

The parameter ν values that we have played with correspond to the weakest nontrivial "turbulence", and it is an open question to what extent the approach remains implementable as the system goes more turbulent.

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[16] R.F.S. Andrade and A. Rauh, *"The Lorenz model and the method of Carleman embedding"*, *Phys. Lett.* **A 82**, 276 (1981).
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[17] W.H. Steeb, *"Embedding of nonlinear finite dimensional systems in linear infinite dimensional systems and Bose operators"*, *Hadronic Journal* **6** 68, (1983).
Nonlinear autonomous systems x=f(x), where f is a polynomial in x, can be embedded into an infinite linear system with a technique described by Carleman (1932). The author demonstrates that the infinite linear system can be written with the help of Bose annihilation and creation operators

[18] K. Kowalski, *"Hilbert space description of classical dynamical systems"*, *Physica* **A 145**, 408 (1987).
Classical dynamical systems can be described by a Hilbert space, Schrodinger-like equation with the Hamiltonian (a non-Hermitian one) expressed in terms of Bose creation and annihilation operators. The presented formalism includes the Carleman embedding as the special case.

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Nonlinear dynamical systems with analytic nonlinearities can be brought down to the abstract Schrodinger equation in Hilbert space with boson Hamiltonian. The Fourier coefficients of the expansion of solutions to the Schrodinger equation in the particular occupation number representation are expressed by means of the classical orthogonal polynomials. The introduced formalism amounts to a generalization of the classical methods for linearization of nonlinear differential equations such as the Carleman embedding technique and Koopman approach