Appendix G

Finding cycles

(C. Chandre)

G.1 Newton-Raphson method

G.1.1 Contraction rate

Consider a d-dimensional map $x' = f(x)$ with an unstable fixed point x_*. The Newton-Raphson algorithm is obtained by iterating the following map

$$x' = g(x) = x - (J(x) - 1)^{-1} (f(x) - x) .$$

The linearization of g near x_* leads to

$$x_* + \epsilon' = x_* + \epsilon - (J(x_*) - \mathbf{1})^{-1} (f(x_*) + J(x_*)\epsilon - x_* - \epsilon) + O\left(\|\epsilon\|^2\right) ,$$

where $\epsilon = x - x_*$. Therefore,

$$x' - x_* = O\left((x - x_*)^2\right) .$$

After n steps and if the initial guess x_0 is close to x_*, the error decreases super-exponentially

$$g^n(x_0) - x_* = O\left((x_0 - x_*)^{2^n}\right) .$$

G.1.2 Computation of the inverse

The Newton-Raphson method for finding n-cycles of d-dimensional mappings using the multi-shooting method reduces to the following equation

$$\begin{pmatrix} \mathbf{1} & & & & -Df(x_n) \\ -Df(x_1) & \mathbf{1} & & & \\ & \cdots & \mathbf{1} & & \\ & & -Df(x_{n-1}) & \mathbf{1} & \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \cdots \\ \delta_n \end{pmatrix} = - \begin{pmatrix} F_1 \\ F_2 \\ \cdots \\ F_n \end{pmatrix}, \quad (G.1)$$

where $Df(x)$ is the $[d \times d]$ Jacobian matrix of the map evaluated at the point x, and $\delta_m = x'_m - x_m$ and $F_m = x_m - f(x_{m-1})$ are d-dimensional vectors. By some starightforward algebra, the vectors δ_m are expressed as functions of the vectors F_m:

$$\delta_m = -\sum_{k=1}^{m} \beta_{k,m-1} F_k - \beta_{1,m-1} \left(\mathbf{1} - \beta_{1,n}\right)^{-1} \left(\sum_{k=1}^{n} \beta_{k,n} F_k \right), \qquad (G.2)$$

for $m = 1, \ldots, n$, where $\beta_{k,m} = Df(x_m) Df(x_{m-1}) \cdots Df(x_k)$ for $k < m$ and $\beta_{k,m} = \mathbf{1}$ for $k \geq m$. Therefore, finding n-cycles by a Newton-Raphson method with multiple shooting requires the inversing of a $[d \times d]$ matrix $\mathbf{1} - Df(x_n) Df(x_{n-1}) \cdots Df(x_1)$.

G.2 Hybrid Newton-Raphson / relaxation method

Consider a d-dimensional map $x' = f(x)$ with an unstable fixed point x_*. The transformed map is the following one:

$$x' = g(x) = x + \gamma C(f(x) - x),$$

where $\gamma > 0$ and C is a $d \times d$ invertible constant matrix. We notice that x_* is also a fixed point of g. Consider the stability matrix at the fixed point x_*

$$A_g = \left.\frac{dg}{dx}\right|_{x=x_*} = 1 + \gamma C(A_f - 1).$$

The matrix C is constructed such that the eigenvalues of A_g are of modulus less than one. Assume that A_f is diagonalizable: In the basis of diagonalization, the matrix writes:

$$\tilde{A}_g = 1 + \gamma \tilde{C}(\tilde{A}_f - 1),$$

where $\tilde{A}_f$ is diagonal with elements μ_i. We restrict the set of matrices $\tilde{C}$ to diagonal matrices with $\tilde{C}_{ii} = \epsilon_i$ where $\epsilon_i = \pm 1$. Thus $\tilde{A}_g$ is diagonal with eigenvalues $\gamma_i = 1 + \gamma\epsilon_i(\mu_i - 1)$. The choice of γ and ϵ_i is such that $|\gamma_i| < 1$. It is easy to see that if $\mathrm{Re}(\mu_i) < 1$ one has to choose $\epsilon_i = 1$, and if $\mathrm{Re}(\mu_i) > 1$, $\epsilon_i = -1$. If λ is chosen such that

$$0 < \gamma < \min_{i=1,\dots,d} \frac{2|\mathrm{Re}(\mu_i) - 1|}{|\mu_i - 1|^2},$$

all the eigenvalues of A_g have modulus less that one. The contraction rate at the fixed point for the map g is then $\max_i |1 + \gamma\epsilon_i(\mu_i - 1)|$. We notice that if $\mathrm{Re}(\mu_i) = 1$, it is not possible to stabilize x_* by the set of matrices γC.

From the construction of C, we see that 2^d choices of matrices are possible. For example, for two-dimensional systems, these matrices are

$$C \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

For 2-dimensional dissipative maps, the eigenvalues satisfy $\mathrm{Re}(\mu_1)\mathrm{Re}(\mu_2) \leq \det Df < 1$. The case $(\mathrm{Re}(\mu_1) > 1, \mathrm{Re}(\mu_2) > 1)$ which is stabilized by $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ has to be discarded. The minimal set is reduced to three matrices.