Appendix B

Infinite-dimensional flows

Flows described by partial differential equations are considered infinite dimensional because if one writes them down as a set of ordinary differential equations (ODEs) then one needs an infinity of the ordinary kind to represent the dynamics of one equation of the partial kind (PDE). Even though the phase space is infinite dimensional, for many systems of physical interest the global attractor is finite dimensional. We illustrate how this works with a concrete example, the Kuramoto-Sivashinsky system.

B.0.1 Partial differential equations

First, a few words about partial differential equations in general. Many of the partial differential equations of mathematical physics can be written in the quasi-linear form

$$\partial_t u = Au + N(u)\,, \tag{B.1}$$

where u is a function (possibly a vector function) of the coordinate x and time t, A is a linear operator, usually containing the Laplacian and a few other derivatives of u, and $N(u)$ is the nonlinear part of the equation (terms like $u\partial_x u$ in (2.14)).

Not all equations are stated in the form (B.1), but they can easily be so transformed, just as the ordinary differential equations can be rewritten as first-order systems. We will illustrate the method with a variant of the D'Alambert's wave equation describing a plucked string:

$$\partial_{tt} y = \left(c + \frac{1}{2}\left(\partial_x y\right)^2\right)\partial_{xx} y\,. \tag{B.2}$$

Were the term $\partial_x y$ small, this equation would be just the ordinary wave equation. To rewrite the equation in the first order form (B.1), we need a field $u = (y, w)$ that is two-dimensional,

$$\partial_t \begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c\partial_{xx} & 0 \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ \partial_{xx} y (\partial_x y)^2/2 \end{bmatrix} . \tag{B.3}$$

The [2×2] matrix is the linear operator A and the vector on the far right is the nonlinear function $N(u)$. Unlike ordinary functions, differentiations are part of the function. The nonlinear part can also be expressed as a function on the infinite set of numbers that represent the field, as exemplified by the Kuramoto-Sivashinsky system (2.14).

The usual technique for solving the linear part is to use Fourier methods. Just as in the ordinary differential equation case, one can integrate the linear part of

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$$\partial_t u = A u \tag{B.4}$$

to obtain

$$u(x, t) = e^{tA} u(x, 0) . \tag{B.5}$$

If u is expressed as Fourier series $\sum_k a_k \exp(ikx)$, as we will do for the Kuramoto-Shivashinsky system, then we can determine the action of e^{tA} on $u(x, 0)$. This can be done because differentiations in A act rather simply on the exponentials. For example,

$$e^{t\partial_x} u(x, 0) = e^{t\partial_x} \sum_k a_k e^{ikx} = \sum_k a_k \frac{(it)^k}{k!} e^{ikx} . \tag{B.6}$$

Depending on the behavior of the linear part, one distinguishes three classes of partial differential equations: *diffusion, wave, and potential.* The classification relies on the solution by a Fourier series, as in (B.5). In mathematical literature these equations are also called *parabolic, hyperbolic* and *elliptic*. If the nonlinear part $N(u)$ is as big as the linear part, the classification is not a good indication of behavior, and one can encounter features of one class of equations while studying the others.

In *diffusion-type* equations the modes of high frequency tend to become smooth, and all initial conditions tend to an attractor, called the *inertial manifold*. The Kuramoto-Sivashinsky system studied below is of this type.

The solution being attracted to the inertial manifold does not mean that the amplitudes of all but a finite number of modes go to zero (alas were we so lucky), but that there is a finite set of modes that could be used to describe any solution of the inertial manifold. The only catch is that there is no simple way to discover what these inertial manifold modes might be.

In *wave-like* equations the high frequency modes do not die out and the solutions tend to be distributions. The equations can be solved by variations on the WKB idea: the wave-like equations can be approximated by the trajectories of the wave fronts.

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Elliptic equations have no time dependence and do not represent dynamical systems.