Appendix C

Stability of Hamiltonian flows

C.1 Symplectic invariance

(M.J. Feigenbaum and P. Cvitanović)

The symplectic structure of Hamilton's equations buys us much more than the incompressibility, or the phase space volume conservation alluded to in sect. 5.1. We assume you are at home with Hamiltonian formalism. If you would like to see the Hamilton's equations derived, Hamilton's original line of reasoning is retraced in sect. 26.1.1. The evolution equations for any p, q dependent quantity $Q = Q(p, q)$ are given by (7.33).

sect. 26.1.1

In terms of the Poisson brackets, the time evolution equation for $Q = Q(p, q)$ is given by (7.35). We now recast the symplectic condition (5.8) in a form convenient for using the symplectic constraints on $\mathbf{J}$. Writing $x(t) = x' = [p', q']$ and the Jacobian matrix and its inverse

$$\mathbf{J} = \begin{pmatrix} \frac{\partial q'}{\partial q} & \frac{\partial q'}{\partial p} \\ \frac{\partial p'}{\partial q} & \frac{\partial p'}{\partial p} \end{pmatrix}, \qquad \mathbf{J}^{-1} = \begin{pmatrix} \frac{\partial q}{\partial q'} & \frac{\partial q}{\partial p'} \\ \frac{\partial p}{\partial q'} & \frac{\partial p}{\partial p'} \end{pmatrix}, \tag{C.1}$$

we can spell out the symplectic invariance condition (5.8):

$$\begin{aligned} \frac{\partial q'_k}{\partial q_i}\frac{\partial p'_k}{\partial q_j} - \frac{\partial p'_k}{\partial q_i}\frac{\partial q'_k}{\partial q_j} &= 0 \\ \frac{\partial q'_k}{\partial p_i}\frac{\partial p'_k}{\partial p_j} - \frac{\partial p'_k}{\partial p_i}\frac{\partial q'_k}{\partial p_j} &= 0 \\ \frac{\partial q'_k}{\partial q_i}\frac{\partial p'_k}{\partial p_j} - \frac{\partial p'_k}{\partial q_i}\frac{\partial q'_k}{\partial p_j} &= \delta_{ij}\,. \end{aligned} \tag{C.2}$$

From (5.9) we obtain

$$\frac{\partial q_i}{\partial q'_j} = \frac{\partial p'_j}{\partial p_i}, \quad \frac{\partial p_i}{\partial p'_j} = \frac{\partial q'_j}{\partial q_i}, \quad \frac{\partial q_i}{\partial p'_j} = -\frac{\partial q'_j}{\partial p_i}, \quad \frac{\partial p_i}{\partial q'_j} = -\frac{\partial p'_j}{\partial q_i}. \tag{C.3}$$

Taken together, (C.3) and (C.2) imply that the flow conserves the $[p, q]$ Poisson brackets

$$\begin{aligned} [q_i, q_j] &= \frac{\partial q_i}{\partial p'_k}\frac{\partial q_j}{\partial q'_k} - \frac{\partial q_j}{\partial p'_k}\frac{\partial q_i}{\partial q'_k} = 0 \\ [p_i, p_j] &= 0\,, \qquad [p_i, q_j] = \delta_{ij}\,, \end{aligned} \tag{C.4}$$

that is, the transformations induced by a Hamiltonian flow are *canonical*, preserving the form of the equations of motion. The first two relations are symmetric under i, j interchange and yield $D(D-1)/2$ constraints each; the last relation yields D^2 constraints. Hence only $(2D)^2 - 2D(D-1)/2 - D^2 = 2D^2 + D$ elements of $\mathbf{J}$ are linearly independent, as it behooves group elements of the symplectic group $Sp(2D)$.

We have now succeeded in making the full set of constraints explicit - as we shall see in appendix D, this will enable us to implement dynamics in such a way that the symplectic invariance will be automatically preserved.

Consider the symplectic product of two infinitesimal vectors

$$\begin{aligned} (\delta x, \delta \hat{x}) &= \delta x^T \omega \delta \hat{x} = \delta p_i \delta \hat{q}_i - \delta q_i \delta \hat{p}_i \\ &= \sum_{i=1}^{D} \{\text{oriented area in the } (p_i, q_i) \text{ plane}\}\,. \end{aligned} \tag{C.5}$$

Time t later we have

$$(\delta x', \delta \hat{x}') = \delta x^T \mathbf{J}^T \omega \mathbf{J} \delta \hat{x} = \delta x^T \omega \delta \hat{x}\,.$$

This has the following geometrical meaning. We imagine there is a reference phase space point. We then define two other points infinitesimally close so that the vectors δx and $\delta \hat{x}$ describe their displacements relative to the reference point. Under the dynamics, the three points are mapped to three new points which are still infinitesimally close to one another. The meaning of the above expression is that the symplectic area spanned by the three final points is the same as that spanned by the inital points. The integral (Stokes theorem) version of this infinitesimal area invariance states that for

Hamiltonian flows the D oriented areas $\mathcal{V}_i$ bounded by D loops $\Omega\mathcal{V}_i$, one per each (p_i, q_i) plane, are separately conserved:

$$\int_{\mathcal{V}} dp \wedge dq = \oint_{\Omega\mathcal{V}} p \cdot dq = \text{invariant} . \tag{C.6}$$

C.2 Monodromy matrix for Hamiltonian flows

(G. Tanner)

It is not the Jacobian matrix of the flow, but the *monodromy* matrix, which enters the trace formula. This matrix gives the time dependence of a displacement perpendicular to the flow on the energy manifold. Indeed, we discover some trivial parts in the Jacobian matrix $\mathbf{J}$. An initial displacement in the direction of the flow $x = \omega \nabla H(x)$ transfers according to $\delta x(t) = x_t(t)\delta t$ with δt time independent. The projection of any displacement on δx on $\nabla H(x)$ is constant, i.e. $\nabla H(x(t))\delta x(t) = \delta E$. We get the equations of motion for the monodromy matrix directly choosing a suitable local coordinate system on the orbit $x(t)$ in form of the (non singular) transformation $\mathbf{U}(x(t))$:

$$\tilde{\mathbf{J}}(x(t)) = \mathbf{U}^{-1}(x(t))\, \mathbf{J}(x(t))\, \mathbf{U}(x(0)) \tag{C.7}$$

These lead to

$$\begin{aligned} & \dot{\tilde{\mathbf{J}}} &=& \tilde{\mathbf{L}}\, \tilde{\mathbf{J}} \\ \text{with} \quad & \tilde{\mathbf{L}} &=& \mathbf{U}^{-1}(\mathbf{L}\mathbf{U} - \dot{\mathbf{U}}) \end{aligned} \tag{C.8}$$

Note that the properties a) – c) are only fulfilled for $\tilde{\mathbf{J}}$ and $\tilde{\mathbf{L}}$, if $\mathbf{U}$ itself is symplectic.
Choosing $x_E = \nabla H(t)/|\nabla H(t)|^2$ and x_t as local coordinates uncovers the two trivial eigenvalues 1 of the transformed matrix in (C.7) at any time t.
Setting $\mathbf{U} = (x_t^T, x_E^T, x_1^T, \ldots, x_{2d-2}^T)$ gives

$$\tilde{\mathbf{J}} = \begin{pmatrix} 1 & * & * & \ldots & * \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & * & & & \\ \vdots & \vdots & & \mathbf{m} & \\ 0 & * & & & \end{pmatrix} ; \qquad \tilde{\mathbf{L}} = \begin{pmatrix} 0 & * & * & \ldots & * \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & * & & & \\ \vdots & \vdots & & \mathbf{l} & \\ 0 & * & & & \end{pmatrix} , \tag{C.9}$$

The matrix $\mathbf{m}$ is now the monodromy matrix and the equation of motion are given by

$$\dot{\mathbf{m}} = \mathbf{l}\, \mathbf{m}. \tag{C.10}$$

The vectors $x_1, \ldots, x_{2d-2}$ must span the space perpendicular to the flow on the energy manifold.

For a system with two degrees of freedom, the matrix $\mathbf{U}(\mathbf{t})$ can be written down explicitly, i.e.

$$\mathbf{U}(t) = (x_t, x_1, x_E, x_2) = \begin{pmatrix} \dot{x} & -\dot{y} & -\dot{u}/q^2 & -\dot{v}/q^2 \\ \dot{y} & \dot{x} & -\dot{v}/q^2 & \dot{u}/q^2 \\ \dot{u} & \dot{v} & \dot{x}/q^2 & -\dot{y}/q^2 \\ \dot{v} & -\dot{u} & \dot{y}/q^2 & \dot{x}/q^2 \end{pmatrix} \tag{C.11}$$

with $x^T = (x, y; u, v)$ and $q = |\nabla H| = |\dot{x}|$. The matrix $\mathbf{U}$ is non singular and symplectic at every phase space point x (except the equilibrium points $\dot{x} = 0$). The matrix elements for $\mathbf{l}$ are given (C.13). One distinguishes 4 classes of eigenvalues of $\mathbf{m}$.

- *stable* or *elliptic*, if $\Lambda = e^{\pm i\pi\nu}$ and $\nu \in]0, 1[$.
- *marginal*, if $\Lambda = \pm 1$.
- *hyperbolic*, *inverse hyperbolic*, if $\Lambda = e^{\pm\lambda}$, $\Lambda = -e^{\pm\lambda}$; $\lambda > 0$ is called the Lyapunov exponent of the periodic orbit.
- *loxodromic*, if $\Lambda = e^{\pm u \pm i\Psi}$ with u and Ψ real. This is the most general case possible only in systems with 3 or more degree of freedoms.

For 2 degrees of freedom, i.e. $\mathbf{m}$ is a (2×2) matrix, the eigenvalues are determined by

$$\lambda = \frac{\mathrm{Tr}(\mathbf{m}) \pm \sqrt{\mathrm{Tr}(\mathbf{m})^2 - 4}}{2}, \tag{C.12}$$

i.e. $\mathrm{Tr}(\mathbf{m}) = 2$ separates stable and unstable behavior.

The $\mathbf{l}$ matrix elements for the local transformation (C.11) are

$$\tilde{\mathbf{l}}_{11} \;=\; \frac{1}{q}[(h_x^2 - h_y^2 - h_u^2 + h_v^2)(h_{xu} - h_{yv}) + 2(h_x h_y - h_u h_v)(h_{xv} + h_{yu})$$

$$
\begin{aligned}
& -(h_x h_u + h_y h_v)(h_{xx} + h_{yy} - h_{uu} - h_{vv})] \\
\tilde{\mathrm{l}}_{12} &= \frac{1}{q^2}[(h_x^2 + h_v^2)(h_{yy} + h_{uu}) + (h_y^2 + h_u^2)(h_{xx} + h_{vv}) \\
& -2(h_x h_u + h_y h_v)(h_{xu} + h_{yv}) - 2(h_x h_y - h_u h_v)(h_{xy} - h_{uv})] \\
\tilde{\mathrm{l}}_{21} &= -(h_x^2 + h_y^2)(h_{uu} + h_{vv}) - (h_u^2 + h_v^2)(h_{xx} + h_{yy}) \\
& +2(h_x h_u - h_y h_v)(h_{xu} - h_{yv}) + 2(h_x h_v + h_y h_u)(h_{xv} + h_{yu}) \\
\tilde{\mathrm{l}}_{22} &= -\tilde{\mathrm{l}}_{11}, \qquad \text{(C.13)}
\end{aligned}
$$

with h_i, h_{ij} is the derivative of the Hamiltonian H with respect to the phase space coordinates and $q = |\nabla H|^2$.

Remark C.1 The sign convention of the Poisson bracket. The Poisson bracket is antisymmetric in its arguments and there is a freedom to define it with the opposite sign convention. When such freedoms exist in physics, it is certain that both conventions are in use and this is no exception. In several texts you will see the right hand side of (7.34) defined as $[B, A]$ so that (7.35) is $\frac{dQ}{dt} = [Q, H]$. As long as one is consistent, there should be no problem.

Remark C.2 The sign convention of ω. The overall sign of ω, the symplectic invariant in (5.4), is set by the convention that the Hamilton's principal function (for energy conserving flows) is given by $R(q, q', t) = \int_q^{q'} p_i dq_i - Et$. With this sign convention the action along a classical path is minimal, and the kinetic energy of a free particle is positive.