Appendix I

Discrete symmetries

I.1 Preliminaries and definitions

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In the following we will define what we mean by the concepts *group*, *representation*, *symmetry of a dynamical system*, and *invariance*.

Group axioms. First, we define a group in abstract terms: A group G is a set of elements $g_1, g_2, g_3, \ldots$ for which a law of *composition* or *group multiplication* is given such that the product $g_2 \circ g_1$ (which we will also just abbreviate as $g_2 g_1$) of any two elements satisfies the following conditions:

1. If $g_1, g_2 \in G$, then $g_2 \circ g_1 \in G$.
2. The group multiplication is associative: $g_3 \circ (g_2 \circ g_1) = (g_3 \circ g_2) \circ g_1$.
3. The group G contains an element e called *identity* such that $g \circ e = e \circ g = g$ for every element $g \in G$.
4. For every element $g \in G$, there exists an unique element $h \in G$ such that $h \circ g = g \circ h = e$. The element h is called *inverse* of g, and is denoted by $h = g^{-1}$.

A *finite* group is a group with a finite number of elements

$$G = \{e, g_2, \ldots, g_{|G|}\} ,$$

where $|G|$, the number of elements, will be referred to as *order* of the group.

Matrix group on vector space. We will now apply these abstract group definitions to the set of $[d \times d]$-dimensional non-singular matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ acting in a d-dimensional vector space $V \in \mathbb{R}^d$, that is the product of matrices $\mathbf{A}$ and $\mathbf{B}$ gives the single matrix $\mathbf{C}$, such that

$$\mathbf{C}v = \mathbf{B}(\mathbf{A}v) \in V, \qquad \forall v \in V. \tag{I.1}$$

The identity of the group is the unit matrix $\mathbf{1}$ which leaves all vectors in V unchanged. Every matrix in the group has a unique inverse.

Linear representation of a group. Let us now map the abstract group G *homeomorphically* on a group of matrices $\mathsf{D}(G)$ in the vector space V, that isin such a way that the group properties, especially the group multiplication, are preserved:

1. Any $g \in G$ is mapped to a matrix $\mathbf{D}(g) \in \mathsf{D}(G)$.
2. The group product $g_2 \circ g_1 \in G$ is mapped onto the matrix product $\mathbf{D}(g_2 \circ g_1) = \mathbf{D}(g_2)\mathbf{D}(g_1)$.
3. The associativity is preserved: $\mathbf{D}(g_3 \circ (g_2 \circ g_1)) = \mathbf{D}(g_3)(\mathbf{D}(g_2)\mathbf{D}(g_1)) = (\mathbf{D}(g_3)(\mathbf{D}(g_2))\mathbf{D}(g_1)$.
4. The identity element $e \in G$ is mapped onto the unit matrix $\mathsf{D}(e) = \mathbf{1}$ and the inverse element $g^{-1} \in G$ is mapped onto the inverse matrix $\mathbf{D}(g^{-1}) = [\mathbf{D}(g)]^{-1} \equiv \mathbf{D}^{-1}(g)$.

We call the so defined matrix group $\mathsf{D}(G)$ a linear or matrix *representation* of the group G in the *representation space* V. Note that the matrix operation on a vector is by definition linear. We use the specification *linear* in order to discriminate the matrix representations from other operator representations that do not have to be linear, in general. Throughout this appendix we only consider linear representations.

If the dimensionality of V is d, we say the representation is an *d-dimensional representation* or has the *degree d*. The matrices $\mathbf{D}(g) \in \mathsf{D}(G)$ are non-singular $[d \times d]$ matrices, which we will also just abbreviate as $\mathbf{g}$, that is$x' = \mathbf{g}x$ corresponds to the normal matrix operation $x'_i = \sum_{j=1}^{d} (\mathbf{g})_{ij} x_j = \sum_{j=1}^{d} g_{ij} x_j$.

Character of a representation. The character of $\chi_\alpha(g)$ of an d-dimensional representation $\mathbf{D}(g)$ of the group element g of a discrete group G is defined as trace

$$\chi_\alpha(g) = \sum_{i=1}^{n} \mathbf{D}_{ii}(g) \equiv \operatorname{tr} \mathbf{D}(g) \,.$$

Note especially that $\chi(e) = n$, since $\mathbf{D}_{ij}(e) = \delta_{ij}$ for $1 \le i, j \le n$.

Faithful representations. If the homomorphismus mapping G on $\mathsf{D}(G)$ becomes an isomorphism, the representation is said to be *faithful*. In this case the order of the group of matrices $D(G)$ is equal to the order $|G|$ of the group. In general, however, there will be several elements of G that will be mapped on the unit matrix $\mathbf{D}(e) = \mathbf{1}$. This property can be used to define a subgroup $H \subset G$ of the group G consisting of all elements $h \in G$ that are mapped to the unit matrix of a given representation. Then the cosidered representation is a faithful representation of the *factor group* G/H.

Equivalent representations. From this remarks it should be clear that the representation of a group is by no means unique. If the basis in the d-dimensional vector space V is changed, the matrices $\mathbf{D}(g)$ have to be replaced by their transformations $\mathbf{D}'(g)$. In this case, however, the new matrices $\mathbf{D}'(g)$ and the old matrices $\mathbf{D}(g)$ are related by an equivalence transformation through a non-singular matrix $\mathbf{C}$

$$\mathbf{D}'(g) = \mathbf{C}\,\mathbf{D}(g)\,\mathbf{C}^{-1} \,.$$

Thus, the group of matrices $\mathbf{D}'(g)$ form an equivalent representation $\mathsf{D}'(G)$ to the representation $\mathsf{D}(G)$ of the group G. The equivalent representations have the same structure, although the matrices look different. Because of the cylic nature of the trace and because equivalent representations have the same dimension, the character of equivalent representations is the same

$$\chi(g) = \sum_{i=1}^{n} \mathbf{D}'_{ii}(g) = \operatorname{tr}\left(\mathbf{D}'(g)\right) = \operatorname{tr}\left(\mathbf{C}\mathbf{D}(g)\mathbf{C}^{-1}\right) \,.$$

Regular representation of a group. The *regular* representation of a group is a special representation that is defined as follows: If we define the elements of a finite group as $g_1, g_2, \ldots, g_{|G|}$, the multiplying from the left by any element g_ν permutes the $g_1, g_2, \ldots, g_{|G|}$ among themselves. We

can represent the element g_ν by the permutations of the $|G|$ "coordinates" $g_1, g_2, \ldots, g_{|G|}$. Thus for $i, j = 1, \ldots, |G|$, we define the *regular representation*

$$\mathbf{D}_{ij}(g_\nu) = \begin{cases} \delta_{jl_i} & \text{if } g_\nu g_i = g_{l_i} \text{ with } l_i = 1, \ldots, |G| \,, \\ 0 & \text{otherwise} \,. \end{cases}$$

In this regular representation the diagonal elements of all matrices are zero except for the element g_{ν_0} with $g_{\nu_0} g_i = g_i$, that isfor the identity element e. So in the regular representation the character is given by

$$\chi(g) = \begin{cases} 1 & \text{for} \quad g = e \,, \\ 0 & \text{for} \quad g \neq e \,. \end{cases}$$

Passive and active coordinate transformations. We have to discriminate between *active* and *passive* coordinate transformations. An active (coordinate) transformation corresponds to an non-singular $d \times d$ matrix that actively shifts/changes the vector $x \in \mathcal{M}$

$$x \to \mathbf{T}x.$$

The corresponding passive coordinate transformation changes the coordinate system with respect to which the vector $f(x) \in \mathcal{M}$ is measured. Thus it is given by

$$f(x) \to \mathbf{T}^{-1} f(x) = f(\mathbf{T}^{-1} x).$$

Note that the combination of an passive and active coordinate transformation results to the identity

$$f(x) = \mathbf{T}^{-1} f(\mathbf{T}x) \,.$$

On the other hand, the evolution operator $\mathcal{L}(x, y)$ satisfies the following identity

$$\mathcal{L}(x, y) = \left| \det \left(\frac{\partial \mathbf{T}x}{\partial x} \right) \right| \mathcal{L}(\mathbf{T}x, \mathbf{T}y) = |\det \mathbf{T}|\, \mathcal{L}(\mathbf{T}x, \mathbf{T}y).$$

Note the appearance of $\det \mathbf{T}$ instead of $\det \mathbf{T}^{-1}$ and therefore the contravariant transformation property of $\mathcal{L}(x, y)$ in correspondence to maps

$f(x)$. If the coordinate transformation $\mathbf{T}$ belongs to the linear non-singular representation of a *discrete* (that isfinite) symmetry group G, then $|\det \mathbf{T}| = 1$, since for any element g of a finite group G, where exists a number m such that

$$g^n \equiv \underbrace{g \circ g \circ \ldots \circ g}_{m \text{ times}} = e.$$

Thus $\mathbf{T}$ corresponds to the mth root of $\mathbf{1}$ and the modulus of its determinant is unity.

Symmetry of dynamical system. A dynamical system $(\mathcal{M}, f)$ is invariant under a discrete symmetry group $G = \{e, g_2, \ldots, g_{|G|}\}$, if the map $f : \mathcal{M} \to \mathcal{M}$ (or the continous flow f^t) from the d-dimensional manifold $\mathcal{M}$ into itself (with d finite) is invariant

$$f(\mathbf{g}x) = \mathbf{g}f(x)$$

for *any* coordinate $x \in \mathcal{M}$ and *any* finite non-singular linear representation (that isa non-singular $d \times d$ matrix) $\mathbf{g}$ of any element $g \in G$. So a *symmetry for a dynamical system* $(\mathcal{M}, f)$ has to satisfy the two conditions

$$\begin{array}{ll} 1) & \mathbf{g}x \in \mathcal{M} \quad \forall x \in \mathcal{M} \text{ and } \forall g \in G\,, \\ 2) & [\mathbf{D}(g), f] = 0 \quad \forall f : \mathcal{M} \to \mathcal{M} \text{ and } \forall g \in G\,. \end{array}$$

Group integration. Note the following laws

$$\frac{1}{|G|} \sum_{g \in G} = 1$$

and therefore

$$\frac{1}{|G|} \sum_{g \in G} \mathbf{D}(g_i) = \mathbf{D}(g_{i_0}), \quad i_0 \text{ fixed}\,.$$

However,

$$\frac{1}{|G|} \sum_{g \in G} \mathbf{D}(g) = \mathbf{0},$$

where $\mathbf{0}$ is the zero matrix of same dimension as the representations $\mathbf{D}(g) \in D(G)$. In particular,

$$\frac{1}{|G|} \sum_{g \in G} \chi_\alpha(g) = \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{d_\alpha} D(g)_{ii} = 0.$$

Furthermore, if we consider all non-equilavent irreducible representations of a group G, then the quantities $D^{(\alpha)}_{ij}(g)$ for fixed α, i and j

Orthonormalitity of characters. But what can we say about

$$\frac{1}{|G|} \sum_{g \in G} \chi_\alpha(hg)\chi_\alpha(g^{-1}k^{-1}) \quad \text{with } h, k \in G \text{ fixed ?}$$

Note the following relation

$$\delta_{ab}\delta_{cd} = \frac{1}{n}\delta_{ad}\delta_{cb} + \left(\delta_{ab}\delta_{cd} - \frac{1}{n}\delta_{ad}\delta_{cb}\right).$$

Projection operators. The projection operator onto the α irreducible subspace of dimension d_α is given by

$$\mathbf{P}_\alpha = \frac{d_\alpha}{|G|} \sum_{g \in G} \chi_\alpha(g)\mathbf{g}^{-1}.$$

Note that $\mathbf{P}_\alpha$ is a $[d \times d]$-dimensional matrix as the representation $\mathbf{g}$.

Irreducible subspaces of the evolution operator.

$$\mathcal{L} = \sum_\alpha \operatorname{tr} \mathcal{L}_\alpha$$

with

$$\mathcal{L}_\alpha(y, x) = \frac{d_\alpha}{|G|} \sum_{g \in G} \chi_\alpha(g)\mathcal{L}(\mathbf{g}^{-1}y, x),$$

where the prefactor d_α reflects the fact that a d_α-dimensional representation occurs d_α times.

I.2 C_{4v} factorization

If an N-disk arrangement has C_N symmetry, and the disk visitation sequence is given by disk labels $\{\epsilon_1\epsilon_2\epsilon_3\ldots\}$, only the relative increments $\rho_i = \epsilon_{i+1} - \epsilon_i \mod N$ matter. Symmetries under reflections across axes increase the group to C_{Nv} and add relations between symbols: $\{\epsilon_i\}$ and $\{N-\epsilon_i\}$ differ only by a reflection. As a consequence of this reflection increments become decrements until the next reflection and vice versa. Consider four equal disks placed on the vertices of a square (fig. I.1a). The symmetry group consists of the identity $\mathbf{e}$, the two reflections σ_x, σ_y across x, y axes, the two diagonal reflections σ_{13}, σ_{24}, and the three rotations C_4, C_2 and C_4^3 by angles $\pi/2$, π and $3\pi/2$. We start by exploiting the C_4 subgroup symmetry in order to replace the absolute labels $\epsilon_i \in \{1, 2, 3, 4\}$ by relative increments $\rho_i \in \{1, 2, 3\}$. By reflection across diagonals, an increment by 3 is equivalent to an increment by 1 and a reflection; this new symbol will be called $\underline{1}$. Our convention will be to first perform the increment and then to change the orientation due to the reflection. As an example, consider the fundamental domain cycle 112. Taking the disk 1 → disk 2 segment as the starting segment, this symbol string is mapped into the disk visitation sequence $1_{+1}2_{+1}3_{+2}1\ldots = \overline{123}$, where the subscript indicates the increments (or decrements) between neighboring symbols; the period of the cycle $\overline{112}$ is thus 3 in both the fundamental domain and the full space. Similarly, the cycle $\overline{\underline{1}12}$ will be mapped into $1_{+1}2_{-1}1_{-2}3_{-1}2_{+1}3_{+2}1 = \overline{121323}$ (note that the fundamental domain symbol $\underline{1}$ corresponds to a flip in orientation after the second and fifth symbols); this time the period in the full space is twice that of the fundamental domain. In particular, the fundamental domain fixed points correspond to the following 4-disk cycles:

4-disk		reduced
12	↔	$\underline{1}$
1234	↔	1
13	↔	2

Conversions for all periodic orbits of reduced symbol period less than 5 are listed in table I.1.

This symbolic dynamics is closely related to the group-theoretic structure of the dynamics: the global 4-disk trajectory can be generated by mapping the fundamental domain trajectories onto the full 4-disk space by the accumulated product of the C_{4v} group elements $g_1 = C$, $g_2 = C^2$, $g_{\underline{1}} = \sigma_{diag}C = \sigma_{axis}$, where C is a rotation by $\pi/2$. In the $\overline{\underline{1}12}$ example worked out above, this yields $g_{\underline{1}12} = g_2 g_1 g_{\underline{1}} = C^2 C\sigma_{axis} = \sigma_{diag}$, listed in the last column of table I.1. Our convention is to multiply group elements in

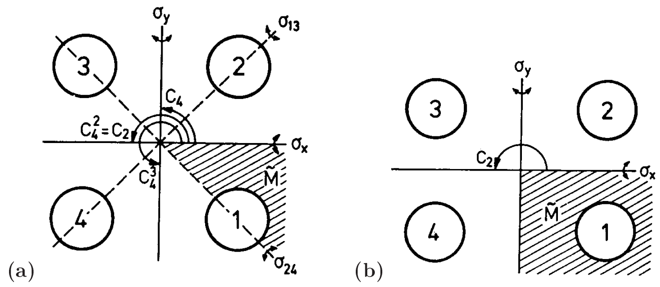

Figure I.1: (a) The symmetries of four disks on a square. (b) The symmetries of four disks on a rectangle. The fundamental domains are indicated by the shaded wedges.

$\tilde{p}$	p	$\mathbf{h}_{\tilde{p}}$
0	1 2	σ_x
1	1 2 3 4	C_4
2	1 3	C_2, σ_{13}
01	12 14	σ_{24}
02	12 43	σ_y
12	12 41 34 23	C_4^3
001	121 232 343 414	C_4
002	121 343	C_2
011	121 434	σ_y
012	121 323	σ_{13}
021	124 324	σ_{13}
022	124 213	σ_x
112	123	e
122	124 231 342 413	C_4

$\tilde{p}$	p	$\mathbf{h}_{\tilde{p}}$
0001	1212 1414	σ_{24}
0002	1212 4343	σ_y
0011	1212 3434	C_2
0012	1212 4141 3434 2323	C_4^3
0021 (a)	1213 4142 3431 2324	C_4^3
0022	1213	e
0102 (a)	1214 2321 3432 4143	C_4
0111	1214 3234	σ_{13}
0112 (b)	1214 2123	σ_x
0121 (b)	1213 2124	σ_x
0122	1213 1413	σ_{24}
0211	1243 2134	σ_x
0212	1243 1423	σ_{24}
0221	1242 1424	σ_{24}
0222	1242 4313	σ_y
1112	1234 2341 3412 4123	C_4
1122	1231 3413	C_2
1222	1242 4131 3424 2313	C_4^3

Table I.1: C_{4v} correspondence between the ternary fundamental domain prime cycles $\tilde{p}$ and the full 4-disk {1,2,3,4} labelled cycles p, together with the C_{4v} transformation that maps the end point of the $\tilde{p}$ cycle into an irreducible segment of the p cycle. For typographical convenience, the symbol $\underline{1}$ of sect. I.2 has been replaced by 0, so that the ternary alphabet is $\{0, 1, 2\}$. The degeneracy of the p cycle is $m_p = 8n_{\tilde{p}}/n_p$. Orbit $\overline{2}$ is the sole boundary orbit, invariant both under a rotation by π and a reflection across a diagonal. The two pairs of cycles marked by (a) and (b) are related by time reversal, but cannot be mapped into each other by C_{4v} transformations.

the reverse order with respect to the symbol sequence. We need these group elements for our next step, the dynamical zeta function factorizations.

The C_{4v} group has four one-dimensional representations, either symmetric (A_1) or antisymmetric (A_2) under both types of reflections, or symmetric under one and antisymmetric under the other (B_1, B_2), and a degenerate pair of two-dimensional representations E. Substituting the C_{4v} characters

C_{4v}	A_1	A_2	B_1	B_2	E
e	1	1	1	1	2
C_2	1	1	1	1	-2
C_4, C_4^3	1	1	-1	-1	0
σ_{axes}	1	-1	1	-1	0
σ_{diag}	1	-1	-1	1	0

into (19.15) we obtain:

$$
\begin{array}{rlclllll}
h_{\tilde{p}} & & & A_1 & A_2 & B_1 & B_2 & E \\
e: & (1-t_{\tilde{p}})^8 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}})^4 \\
C_2: & (1-t_{\tilde{p}}^2)^4 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}})^4 \\
C_4, C_4^3: & (1-t_{\tilde{p}}^4)^2 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1+t_{\tilde{p}}^2)^2 \\
\sigma_{axes}: & (1-t_{\tilde{p}}^2)^4 & = & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}}^2)^2 \\
\sigma_{diag}: & (1-t_{\tilde{p}}^2)^4 & = & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}^2)^2
\end{array}
$$

The possible irreducible segment group elements $\mathbf{h}_{\tilde{p}}$ are listed in the first column; σ_{axes} denotes a reflection across either the x-axis or the y-axis, and σ_{diag} denotes a reflection across a diagonal (see fig. I.1a). In addition, degenerate pairs of boundary orbits can run along the symmetry lines in the full space, with the fundamental domain group theory weights $\mathbf{h}_p = (C_2 + \sigma_x)/2$ (axes) and $\mathbf{h}_p = (C_2 + \sigma_{13})/2$ (diagonals) respectively:

$$
\begin{array}{rlcl}
 & & & \quad A_1 \qquad A_2 \qquad B_1 \qquad B_2 \qquad E \\
\text{axes:} & (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}})(1-0t_{\tilde{p}})(1-t_{\tilde{p}})(1-0t_{\tilde{p}})(1+t_{\tilde{p}})^2 \\
\text{diagonals:} & (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}})(1-0t_{\tilde{p}})(1-0t_{\tilde{p}})(1-t_{\tilde{p}})(1+t_{\tilde{p}})^2
\end{array}
\quad (I.2)
$$

(we have assumed that $t_{\tilde{p}}$ does not change sign under reflections across symmetry axes). For the 4-disk arrangement considered here only the diagonal orbits $\overline{13}$, $\overline{24}$ occur; they correspond to the $\overline{2}$ fixed point in the fundamental domain.

The A_1 subspace in C_{4v} cycle expansion is given by

$$
1/\zeta_{A_1} = (1-t_0)(1-t_1)(1-t_2)(1-t_{01})(1-t_{02})(1-t_{12})
$$

$$
\begin{aligned}
& (1-t_{001})(1-t_{002})(1-t_{011})(1-t_{012})(1-t_{021})(1-t_{022})(1-t_{112}) \\
& (1-t_{122})(1-t_{0001})(1-t_{0002})(1-t_{0011})(1-t_{0012})(1-t_{0021})\ldots \\
= \; & 1 - t_0 - t_1 - t_2 - (t_{01} - t_0 t_1) - (t_{02} - t_0 t_2) - (t_{12} - t_1 t_2) \\
& -(t_{001} - t_0 t_{01}) - (t_{002} - t_0 t_{02}) - (t_{011} - t_1 t_{01}) \\
& -(t_{022} - t_2 t_{02}) - (t_{112} - t_1 t_{12}) - (t_{122} - t_2 t_{12}) \\
& -(t_{012} + t_{021} + t_0 t_1 t_2 - t_0 t_{12} - t_1 t_{02} - t_2 t_{01})\ldots
\end{aligned}
\quad (I.3)
$$

(for typographical convenience, $\underline{1}$ is replaced by 0 in the remainder of this section). For one-dimensional representations, the characters can be read off the symbol strings: $\chi_{A_2}(\mathbf{h}_{\tilde{p}}) = (-1)^{n_0}$, $\chi_{B_1}(\mathbf{h}_{\tilde{p}}) = (-1)^{n_1}$, $\chi_{B_2}(\mathbf{h}_{\tilde{p}}) = (-1)^{n_0+n_1}$, where n_0 and n_1 are the number of times symbols 0, 1 appear in the $\tilde{p}$ symbol string. For B_2 all t_p with an odd total number of 0's and 1's change sign:

$$
\begin{aligned}
1/\zeta_{B_2} = \; & (1+t_0)(1+t_1)(1-t_2)(1-t_{01})(1+t_{02})(1+t_{12}) \\
& (1+t_{001})(1-t_{002})(1+t_{011})(1-t_{012})(1-t_{021})(1+t_{022})(1-t_{112}) \\
& (1+t_{122})(1-t_{0001})(1+t_{0002})(1-t_{0011})(1+t_{0012})(1+t_{0021})\ldots \\
= \; & 1 + t_0 + t_1 - t_2 - (t_{01} - t_0 t_1) + (t_{02} - t_0 t_2) + (t_{12} - t_1 t_2) \\
& +(t_{001} - t_0 t_{01}) - (t_{002} - t_0 t_{02}) + (t_{011} - t_1 t_{01}) \\
& +(t_{022} - t_2 t_{02}) - (t_{112} - t_1 t_{12}) + (t_{122} - t_2 t_{12}) \\
& -(t_{012} + t_{021} + t_0 t_1 t_2 - t_0 t_{12} - t_1 t_{02} - t_2 t_{01})\ldots
\end{aligned}
\quad (I.4)
$$

The form of the remaining cycle expansions depends crucially on the special role played by the boundary orbits: by (I.2) the orbit t_2 does not contribute to A_2 and B_1,

$$
\begin{aligned}
1/\zeta_{A_2} = \; & (1+t_0)(1-t_1)(1+t_{01})(1+t_{02})(1-t_{12}) \\
& (1-t_{001})(1-t_{002})(1+t_{011})(1+t_{012})(1+t_{021})(1+t_{022})(1-t_{112}) \\
& (1-t_{122})(1+t_{0001})(1+t_{0002})(1-t_{0011})(1-t_{0012})(1-t_{0021})\ldots \\
= \; & 1 + t_0 - t_1 + (t_{01} - t_0 t_1) + t_{02} - t_{12} \\
& -(t_{001} - t_0 t_{01}) - (t_{002} - t_0 t_{02}) + (t_{011} - t_1 t_{01}) \\
& +t_{022} - t_{122} - (t_{112} - t_1 t_{12}) + (t_{012} + t_{021} - t_0 t_{12} - t_1 t_{02})\ldots
\end{aligned}
\quad (I.5)
$$

and

$$
\begin{aligned}
1/\zeta_{B_1} = \; & (1-t_0)(1+t_1)(1+t_{01})(1-t_{02})(1+t_{12}) \\
& (1+t_{001})(1-t_{002})(1-t_{011})(1+t_{012})(1+t_{021})(1-t_{022})(1-t_{112}) \\
& (1+t_{122})(1+t_{0001})(1-t_{0002})(1-t_{0011})(1+t_{0012})(1+t_{0021})\ldots \\
= \; & 1 - t_0 + t_1 + (t_{01} - t_0 t_1) - t_{02} + t_{12} \\
& +(t_{001} - t_0 t_{01}) - (t_{002} - t_0 t_{02}) - (t_{011} - t_1 t_{01}) \\
& -t_{022} + t_{122} - (t_{112} - t_1 t_{12}) + (t_{012} + t_{021} - t_0 t_{12} - t_1 t_{02})\ldots
\end{aligned}
\quad (I.6)
$$

In the above we have assumed that t_2 does not change sign under C_{4v} reflections. For the mixed-symmetry subspace E the curvature expansion is given by

$$
\begin{aligned}
1/\zeta_E \;=\; & 1 + t_2 + (-{t_0}^2 + {t_1}^2) + (2t_{002} - t_2{t_0}^2 - 2t_{112} + t_2{t_1}^2) \\
& +(2t_{0011} - 2t_{0022} + 2t_2t_{002} - {t_{01}}^2 - {t_{02}}^2 + 2t_{1122} - 2t_2t_{112} \\
& +{t_{12}}^2 - {t_0}^2{t_1}^2) + (2t_{00002} - 2t_{00112} + 2t_2t_{0011} - 2t_{00121} - 2t_{00211} \\
& +2t_{00222} - 2t_2t_{0022} + 2t_{01012} + 2t_{01021} - 2t_{01102} - t_2{t_{01}}^2 + 2t_{02022} \\
& -t_2{t_{02}}^2 + 2t_{11112} - 2t_{11222} + 2t_2t_{1122} - 2t_{12122} + t_2{t_{12}}^2 - t_2{t_0}^2{t_1}^2 \\
& +2t_{002}(-{t_0}^2 + {t_1}^2) - 2t_{112}(-{t_0}^2 + {t_1}^2)) \qquad \text{(I.7)}
\end{aligned}
$$

A quick test of the $\zeta = \zeta_{A_1}\zeta_{A_2}\zeta_{B_1}\zeta_{B_2}\zeta_E^2$ factorization is afforded by the topological polynomial; substituting $t_p = z^{n_p}$ into the expansion yields

$$1/\zeta_{A_1} = 1 - 3z \;, \quad 1/\zeta_{A_2} = 1/\zeta_{B_1} = 1 \;, \quad 1/\zeta_{B_2} = 1/\zeta_E = 1 + z \;,$$

in agreement with (10.42).

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Remark I.1 Labelling conventions While there is a variety of labelling conventions [20.15, 19.13] for the reduced C_{4v} dynamics, we prefer the one introduced here because of its close relation to the group-theoretic structure of the dynamics: the global 4-disk trajectory can be generated by mapping the fundamental domain trajectories onto the full 4-disk space by the accumulated product of the C_{4v} group elements.

I.3 C_{2v} factorization

An arrangement of four identical disks on the vertices of a rectangle has C_{2v} symmetry (fig. I.1b). C_{2v} consists of $\{e, \sigma_x, \sigma_y, C_2\}$, *i.e.*, the reflections across the symmetry axes and a rotation by π.

This system affords a rather easy visualization of the conversion of a 4-disk dynamics into a fundamental domain symbolic dynamics. An orbit leaving the fundamental domain through one of the axis may be folded back by a reflection on that axis; with these symmetry operations $g_0 = \sigma_x$ and $g_1 = \sigma_y$ we associate labels 1 and 0, respectively. Orbits going to the diagonally opposed disk cross the boundaries of the fundamental domain twice; the product of these two reflections is just $C_2 = \sigma_x\sigma_y$, to

$\tilde{p}$	p	**g**
0	1 4	σ_y
1	1 2	σ_x
2	1 3	C_2
01	14 32	C_2
02	14 23	σ_x
12	12 43	σ_y
001	141 232	σ_x
002	141 323	C_2
011	143 412	σ_y
012	143	e
021	142	e
022	142 413	σ_y
112	121 343	C_2
122	124 213	σ_x

$\tilde{p}$	p	**g**
0001	1414 3232	C_2
0002	1414 2323	σ_x
0011	1412	e
0012	1412 4143	σ_y
0021	1413 4142	σ_y
0022	1413	e
0102	1432 4123	σ_y
0111	1434 3212	C_2
0112	1434 2343	σ_x
0121	1431 2342	σ_x
0122	1431 3213	C_2
0211	1421 2312	σ_x
0212	1421 3243	C_2
0221	1424 3242	C_2
0222	1424 2313	σ_x
1112	1212 4343	σ_y
1122	1213	e
1222	1242 4313	σ_y

Table I.2: C_{2v} correspondence between the ternary $\{0,1,2\}$ fundamental domain prime cycles $\tilde{p}$ and the full 4-disk {1,2,3,4} cycles p, together with the C_{2v} transformation that maps the end point of the $\tilde{p}$ cycle into an irreducible segment of the p cycle. The degeneracy of the p cycle is $m_p = 4n_{\tilde{p}}/n_p$. Note that the 012 and 021 cycles are related by time reversal, but cannot be mapped into each other by C_{2v} transformations. The full space orbit listed here is generated from the symmetry reduced code by the rules given in sect. I.3, starting from disk 1.

which we assign the label 2. For example, a ternary string $0\,0\,1\,0\,2\,0\,1\ldots$ is converted into 12143123..., and the associated group-theory weight is given by $\ldots g_1 g_0 g_2 g_0 g_1 g_0 g_0$.

Short ternary cycles and the corresponding 4-disk cycles are listed in table I.2. Note that already at length three there is a pair of cycles ($012 = 143$ and $021 = 142$) related by time reversal, but *not* by any C_{2v} symmetries.

The above is the complete description of the symbolic dynamics for 4 sufficiently separated equal disks placed at corners of a rectangle. However, if the fundamental domain requires further partitioning, the ternary description is insufficient. For example, in the stadium billiard fundamental domain one has to distinguish between bounces off the straight and the curved sections of the billiard wall; in that case five symbols suffice for constructing the covering symbolic dynamics.

The group C_{2v} has four one-dimensional representations, distinguished by their behavior under axis reflections. The A_1 representation is symmetric with respect to both reflections; the A_2 representation is antisymmetric with respect to both. The B_1 and B_2 representations are symmetric under one and antisymmetric under the other reflection. The character table is

C_{2v}	A_1	A_2	B_1	B_2
e	1	1	1	1
C_2	1	1	−1	−1
σ_x	1	−1	1	−1
σ_y	1	−1	−1	1

Substituted into the factorized determinant (19.14), the contributions of periodic orbits split as follows

$$
\begin{array}{rlclccc}
g_{\tilde{p}} & & & A_1 & A_2 & B_1 & B_2 \\
e: & (1-t_{\tilde{p}})^4 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) \\
C_2: & (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) \\
\sigma_x: & (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) \\
\sigma_y: & (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}})
\end{array}
$$

Cycle expansions follow by substituting cycles and their group theory factors from table I.2. For A_1 all characters are $+1$, and the corresponding cycle expansion is given in (I.3). Similarly, the totally antisymmetric subspace factorization A_2 is given by (I.4), the B_2 factorization of C_{4v}. For B_1 all t_p with an odd total number of 0's and 2's change sign:

$$
\begin{aligned}
1/\zeta_{B_1} &= (1+t_0)(1-t_1)(1+t_2)(1+t_{01})(1-t_{02})(1+t_{12}) \\
&\quad (1-t_{001})(1+t_{002})(1+t_{011})(1-t_{012})(1-t_{021})(1+t_{022})(1+t_{112}) \\
&\quad (1-t_{122})(1+t_{0001})(1-t_{0002})(1-t_{0011})(1+t_{0012})(1+t_{0021})\ldots \\
&= 1+t_0-t_1+t_2+(t_{01}-t_0t_1)-(t_{02}-t_0t_2)+(t_{12}-t_1t_2) \\
&\quad -(t_{001}-t_0t_{01})+(t_{002}-t_0t_{02})+(t_{011}-t_1t_{01}) \\
&\quad +(t_{022}-t_2t_{02})+(t_{112}-t_1t_{12})-(t_{122}-t_2t_{12}) \\
&\quad -(t_{012}+t_{021}+t_0t_1t_2-t_0t_{12}-t_1t_{02}-t_2t_{01})\ldots
\end{aligned}
\tag{I.8}
$$

For B_2 all t_p with an odd total number of 1's and 2's change sign:

$$
\begin{aligned}
1/\zeta_{B_2} &= (1-t_0)(1+t_1)(1+t_2)(1+t_{01})(1+t_{02})(1-t_{12}) \\
&\quad (1+t_{001})(1+t_{002})(1-t_{011})(1-t_{012})(1-t_{021})(1-t_{022})(1+t_{112}) \\
&\quad (1+t_{122})(1+t_{0001})(1+t_{0002})(1-t_{0011})(1-t_{0012})(1-t_{0021})\ldots \\
&= 1-t_0+t_1+t_2+(t_{01}-t_0t_1)+(t_{02}-t_0t_2)-(t_{12}-t_1t_2) \\
&\quad +(t_{001}-t_0t_{01})+(t_{002}-t_0t_{02})-(t_{011}-t_1t_{01}) \\
&\quad -(t_{022}-t_2t_{02})+(t_{112}-t_1t_{12})+(t_{122}-t_2t_{12}) \\
&\quad -(t_{012}+t_{021}+t_0t_1t_2-t_0t_{12}-t_1t_{02}-t_2t_{01})\ldots
\end{aligned}
\tag{I.9}
$$

Note that all of the above cycle expansions group long orbits together with their pseudoorbit shadows, so that the shadowing arguments for convergence still apply.

The topological polynomial factorizes as

$$
\frac{1}{\zeta_{A_1}} = 1-3z \quad , \quad \frac{1}{\zeta_{A_2}} = \frac{1}{\zeta_{B_1}} = \frac{1}{\zeta_{B_2}} = 1+z,
$$

consistent with the 4-disk factorization (10.42).

Remark I.2 C_{2v} symmetry C_{2v} is the symmetry of several systems studied in the literature, such as the stadium billiard [3.17], and the 2-dimensional anisotropic Kepler potential [27.18].

I.4 Hénon map symmetries

We note here a few simple symmetries of the Hénon map (3.10). For $b \neq 0$ the Hénon map is reversible: the backward iteration of (3.11) is given by

$$
x_{n-1} = -\frac{1}{b}(1-ax_n^2-x_{n+1})\,. \tag{I.10}
$$

Hence the time reversal amounts to $b \to 1/b$, $a \to a/b^2$ symmetry in the parameter plane, together with $x \to -x/b$ in the coordinate plane, and there is no need to explore the (a,b) parameter plane outside the strip $b \in \{-1,1\}$. For $b=-1$ the map is orientation and area preserving (see (17.1) below),

$$
x_{n-1} = 1-ax_n^2-x_{n+1}\,, \tag{I.11}
$$

the backward and the forward iteration are the same, and the non–wandering set is symmetric across the $x_{n+1}=x_n$ diagonal. This is one of the simplest models of a Poincaré return map for a Hamiltonian flow. For the orientation reversing $b=1$ case we have

$$
x_{n-1} = 1-ax_n^2+x_{n+1}\,, \tag{I.12}
$$

and the non–wandering set is symmetric across the $x_{n+1}=-x_n$ diagonal.

I.5 Symmetries of the symbol square

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Depending on the type of dynamical system, the symbol square might have a variety of symmetries. Under the time reversal

$$\cdots s_{-2}s_{-1}s_0.s_1s_2s_3\cdots \to \cdots s_3s_2s_1.s_0s_{-1}s_{-2}\cdots$$

the points in the symbol square for an orientation preserving map are symmetric across the diagonal $\gamma = \delta$, and for the orientation reversing case they are symmetric with respect to the $\gamma = 1 - \delta$ diagonal. Consequently the periodic orbits appear either in dual pairs $p = s_1s_2s_3\ldots s_n$, $\overline{p} = s_ns_{n-1}s_{n-2}\ldots s_1$, or are self-dual under time reversal, $S_p = S_{\overline{p}}$. For the orientation preserving case a self-dual cycle of odd period has at least one point on the symmetry diagonal. In particular, all fixed points lie on the symmetry diagonal. Determination of such symmetry lines can be of considerable practical utility, as it reduces some of the periodic orbit searches to 1-dimensional searches.

Remark I.3 Symmetries of the symbol square. For a more detailed discussion of the symbolic dynamics symmetries, see refs. [3.4, 9.38].