Chapter 6

Get straight

A Hamiltonian system is said to be "integrable" if one can find a change of coordinates to an action-angle coordinate frame where the phase space dynamics is described by motion on circles, one circle for each degree of freedom. In the same spirit, a natural description of a hyperbolic, unstable flow would be attained if one found a change of coordinates into a frame where the stable/unstable manifolds are straight lines, and the flow is along hyperbolas. Achieving this globally for anything but a handful of contrived examples is too much to hope for. Still, as we shall now show, we can make some headway on straightening out the flow locally.

Even though such nonlinear coordinate transformations are very important, especially in celestial mechanics, we shall not use them much in what follows, so you can safely skip this chapter. Except, perhaps, you might want to convince yourself that cycle stabilities are indeed metric invariants of flows.

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6.1 Changing coordinates

Problems are handed down to us in many shapes and forms, and they are not always expressed in the most convenient way. In order to simplify a given problem, one may stretch, rotate, bend and mix the coordinates, but in doing so, the vector field will also change. The vector field lives in a (hyper)plane tangent to phase space and changing the coordinates of phase space affects the coordinates of the tangent space as well, in a way that we will now describe.

Denote by h the *conjugation function* which maps the coordinates of the initial phase space $\mathcal{M}$ into the reparametrized phase space $\mathcal{M}' = h(\mathcal{M})$, with a point $x \in \mathcal{M}$ related to a point $y \in \mathcal{M}'$ by $y = h(x)$. The change of coordinates must be one-to-one and span both $\mathcal{M}$ and $\mathcal{M}'$, so given any point y we can go back to $x = h^{-1}(y)$. For smooth flows the reparametrized dynamics should support the same number of derivatives as the initial one. If h is a (piecewise) analytic function, we refer to h as a *smooth conjugacy*.

The evolution rule $g^t(y_0)$ on $\mathcal{M}'$ can be computed from the evolution rule $f^t(x_0)$ on $\mathcal{M}$ by taking the initial point $y_0 \in \mathcal{M}'$, going back to $\mathcal{M}$, evolving, and then mapping the final point $x(t)$ back to $\mathcal{M}'$:

$$y(t) = g^t(y_0) = h \circ f^t \circ h^{-1}(y_0)\,. \tag{6.1}$$

Here "$\circ$" stands for functional composition $h \circ f(x) = h(f(x))$, so (6.1) is a shorthand for $y(t) = h(f^t(h^{-1}(y_0)))$.

The vector field $\dot{x} = v(x)$ in $\mathcal{M}$, locally tangent the flow f^t, is related to the flow by differentiation (2.4) along the trajectory. The vector field $\dot{y} = w(y)$ in $\mathcal{M}'$, locally tangent to g^t follows by the chain rule:

$$\begin{aligned} w(y) &= \frac{dg^t}{dt}(y)\bigg|_{t=0} = \frac{d}{dt}\left(h \circ f^t \circ h^{-1}(y)\right)\bigg|_{t=0} \\ &= h'(h^{-1}(y))v(h^{-1}(y)) = h'(x)v(x)\,. \end{aligned} \tag{6.2}$$

With the indices reinstated, this stands for

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$$w_i(y) = \frac{\partial h_i(x)}{\partial x_j} v_j(x)\,, \qquad y_i = h_i(x)\,. \tag{6.3}$$

Imagine that the phase space is a rubber sheet with the flow lines drawn on it. A coordinate change corresponds to pulling and tugging on the rubber sheet, with h sufficiently smooth to preclude violent and irreversible acts such as cutting, glueing, or self-intersections of the distorted rubber sheet. Trajectories that are closed loops in $\mathcal{M}$ will remain closed loops in the new manifold $\mathcal{M}'$, by their shapes will change. Globally h deforms the rubber sheet in a highly nonlinear manner, but locally it simply rescales and shears the tangent field by $\partial_j h_i$, hence the simple transformation law (6.2) for the velocity fields.

The time itself is a parametrization of points along flow lines, and it can also be reparametrized, $s = s(t)$, with the attendent modification of (6.2). An example is the 2-body collision regularization of the helium Hamiltonian (5.3), to be undertaken in sect. 6.2 below.

6.2 Rectification of flows

A profitable way to exploit invariance of dynamics under smooth conjugacies is to use it to pick out the simplest possible representative of an equivalence class. In general and globally these are just words, as we have no clue how to pick such "canonical" representative, but for smooth flows we can always do it localy and for sufficiently short time, by appealing to the *rectification theorem*, a fundamental theorem of ordinary differential equations. The theorem assures us that there exists a solution (at least for a short time interval) and what the solution looks like. The rectification theorem holds in the neighborhood of points of the vector field $v(x)$ that are not singular, that is, everywhere except for the equilibrium points (2.7), and points at which v is infinite. According to the theorem, in a small neighborhood of a non-singular point there exists a change of coordinates $y = h(x)$ such that $\dot{x} = v(x)$ in the new coordinates takes the standard form

$$\begin{aligned} \dot{y}_1 &= 1 \\ \dot{y}_2 = \dot{y}_3 &= \cdots = \dot{y}_d = 0\,, \end{aligned} \tag{6.4}$$

with unit velocity flow along y_1, and no flow along any of the remaining directions.

Example 6.1 Harmonic oscillator, rectified.

As a simple example of global rectification of a flow consider the harmonic oscillator

$$\dot{q} = p\,, \qquad \dot{p} = -q\,. \tag{6.5}$$

The trajectories $x(t) = (p(t), q(t))$ just go around the origin, so a fair guess is that the system would have a simpler representation in polar coordinates $y = (r, \theta)$:

$$h^{-1} : \left\{ \begin{array}{rcl} q &=& h_1^{-1}(r,\theta) = r\cos\theta \\ p &=& h_2^{-1}(r,\theta) = r\sin\theta \end{array} \right. . \tag{6.6}$$

The Jacobian matrix of the transformation is

$$h' = \begin{bmatrix} \cos\theta & \sin\theta \\ -\dfrac{\sin\theta}{r} & -\dfrac{\cos\theta}{r} \end{bmatrix} \tag{6.7}$$

resulting in (6.2) of rectified form

$$\dot{r} = 0\,, \qquad \dot{\theta} = -1\,. \tag{6.8}$$

In the new coordinates the radial coordinate r is constant, and the angular coordinate θ wraps around a cylinder with constant angular velocity. There is a subtle point in

this change of coordinates: the domain of the map h^{-1} is not the plane $\mathbb{R}^2$, but rather the plane minus the origin. We had mapped a plane into a cylinder, and coordinate transformations should not change the topology of the space in which the dynamics takes place; the coordinate transformation is not defined on the equilibrium point $x = (0, 0)$, or $r = 0$.

Example 6.2 Colinear helium, regularized.

Though very simple in form, the Hamiltonian (5.3) is not the most convenient for numerical investigations of the dynamics of the classical helium system. In the (r_1, r_2) coordinates the potential is singular for $r_i \to 0$ nucleus-electron collisions, with velocity diverging to ∞. These 2-body collisions can be regularized by a rescaling of the time and the coordinates $(r_1, r_2, p_1, p_2) \to (Q_1, Q_2, P_1, P_2)$, in a manner to be described in chapter 29. For the purpose at hand it is sufficient to state the result: In the rescaled coordinates the equations of motion are

$$\begin{aligned} \dot{P}_1 &= 2Q_1 \left[2 - \frac{P_2^2}{8} - Q_2^2 \left(1 + \frac{Q_2^2}{R^4} \right) \right] ; \qquad \dot{Q}_1 = \frac{1}{4} P_1 Q_2^2 \\ \dot{P}_2 &= 2Q_2 \left[2 - \frac{P_1^2}{8} - Q_1^2 \left(1 + \frac{Q_1^2}{R^4} \right) \right] ; \qquad \dot{Q}_2 = \frac{1}{4} P_2 Q_1^2 . \end{aligned} \tag{6.9}$$

where $R = (Q_1^2 + Q_2^2)^{1/2}$. These equations look harder to tackle than the harmonic oscillators that you are familiar with from other learned treatises, and indeed they are. But they are also a typical example of kinds of flows that one works with in practice, and the skill required in finding a good re-coordinatization $h(x)$.

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6.3 Rectification of maps

In sect. 6.2 we had argued that nonlinear coordinate transformations can be profitably employed to simplify the representation of a flow. We shall now apply the same idea to nonlinear maps, and determine a smooth nonlinear change of coordinates that flattens out the vicinity of a fixed point and makes the map *linear* in an open neighborhood. In its simplest form the idea can be implemented only for an isolated nondegenerate fixed point (otherwise are needed in the normal form expansion around the point), and only in a finite neigborhood of a point, as the conjugating function in general has a finite radius of convergence. In sect. 6.3.2 we will extend the method to periodic orbits.

6.3.1 Rectification of a fixed point in one dimension

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Consider a 1-dimensional map $x_{n+1} = f(x_n)$ with a fixed point at $x = 0$, with stability $\Lambda = f'(0)$. If $|\Lambda| \neq 1$, one can determine term-by-term the power series for a smooth conjugation $h(x)$ centered at the fixed point, $h(0) = 0$, that flattens out the neighborhood of the fixed point

$$f(x) = h^{-1}(\Lambda h(x)) \tag{6.10}$$

and replaces the nonlinear map $f(x)$ by a *linear* map $y_{n+1} = \Lambda y_n$.

To compute the conjugation h we use the functional equation $h^{-1}(\Lambda x) = f(h^{-1}(x))$ and the expansions

$$\begin{aligned} f(x) &= \Lambda x + x^2 f_2 + x^3 f_3 + \dots \\ h^{-1}(x) &= x + x^2 h_2 + x^3 h_3 + \dots \ . \end{aligned} \tag{6.11}$$

If $h(x)$ is a conjugation, so is any scaling $h(bx)$ of the function for a real number b. Hence the value of $h'(0)$ is not determined by the functional equation (6.10); it is convenient to set $h'(0) = 1$.

The algebra is not particularly illuminating and best left to computers. In any case, for the time being we will not use much beyond the first, linear term in these expansions.

Here we assume $\Lambda \neq 1$. If the fixed point has first $k-1$ derivatives vanishing, the conjugacy is to the kth *normal form*.

In several dimensions, Λ is replaced by the Jacobian matrix, and one has to check that the eigenvalues $\mathbf{J}$ are non-resonant, that is, there is no integer linear relation between the stability exponents (4.37).

6.3.2 Rectification of a 1-dimensional periodic orbit

Now that we have constructed the conjugation function for a fixed point, we turn to the problem of constructing it for periodic orbits. Each point around the cycle has a differently distorted neighborhood, with differing second and higher order derivatives, so we need to compute a different conjugation function h_a at each cycle point x_a. We expand the map f around each cycle point along the cycle,

$$y_a(\phi) = f_a(\phi) - x_{a+1} = \phi f_{a,1} + \phi^2 f_{a,2} + \dots$$

where x_a is a point on the cycle, $f_a(\phi) = f(x_a+\phi)$ is centered on the periodic orbit, and the index k in $f_{a,k}$ refers to the kth order in the expansion (6.11).

For a periodic orbit the conjugation formula (6.10) generalizes to

$$f_a(\phi) = h_{a+1}^{-1}(f_a'(0)h_a(\phi))\,, \qquad a = 1, 2, \cdots, n\,,$$

point by point. The conjugationg functions h_a are obtained in the same way as before, by equating coefficients of the expansion (6.11), and assuming that the cycle stability $\Lambda = \prod_{a=0}^{n-1} f'(x_a)$ is not marginal, $|\Lambda| \neq 1$. The explicit expressions for h_a in terms of f are obtained by iterating around the whole cycle,

$$f^n(x_a + \phi) = h_a^{-1}(\Lambda h_a(\phi)) + x_a\,. \tag{6.12}$$

☞ remark 6.1

evaluated at each cycle point a. Again we have the freedom to set $h_a'(0) = 1$ for all a.

6.3.3 Repeats of cycles

We have traded in our initial nonlinear map f for a (locally) linear map Λy and an equally complicated conjugation function h. What is gained by rewriting the map f in terms of the conjugacy function h? Once the neighborhood of a fixed point is linearized, the repeats of it are trivialized; from the conjugation formula (6.11) one can compute the derivatives of a function composed with itself r times:

$$f^r(x) = h^{-1}(\Lambda^r h(x))\,.$$

One can already discern the form of the expansion for arbitrary repeats; the answer will depend on the conjugacy function $h(x)$ computed for a *single* repeat, and all the dependence on the repeat number will be carried by factors polynomial in Λ^r, a considerable simplification. The beauty of the idea is difficult to gauge at this stage - an appreciation only sets in when one starts computing perturbative corrections, be it in celestial mechanics (where the method was born), be it the quantum or stochastic corrections to "semiclassical" approximations.

6.4 Smooth conjugacies and cycle stability

In sect. 4.5.2 we have established that for a given flow the cycle stability eigenvalues are intrinsic to a given cycle, independent of the staring

point along the cycle. Now we can prove a much stronger statement; cycle stability eigenvalues are *metric invariants* of the flow, the same in *any* representation of the dynamical system.

That the cycle stability eigenvalues are an invariant property of the given dynamical system follows from elementary considerations of sect. 6.1: If the same dynamics is given by a map f in x coordinates, and a map g in the $y = h(x)$ coordinates, then f and g (or any other good representation) are related by (6.2), a reparametrization and a coordinate transformation $g = h \circ f \circ h^{-1}$. As both f and g are arbitrary representations of the dynamical system, the explicit form of the conjugacy h is of no interest, only the properties invariant under any transformation h are of general import. Furthermore, a good representation should not mutilate the data; h must be a *smooth conjugacy* which maps nearby cycle points of f into nearby cycle points of g. This smoothness guarantees that the cycles are not only topological invariants, but that their linearized neighborhoods are also metrically invariant. For a fixed point $f(x) = x$ of a 1-dimensional map this follows from the chain rule for derivatives,

$$\begin{aligned} g'(y) &= h'(f \circ h^{-1}(y)) f'(h^{-1}(y)) \frac{1}{h'(x)} \\ &= h'(x) f'(x) \frac{1}{h'(x)} = f'(x)\,, \end{aligned} \tag{6.13}$$

and the generalization to the stability eigenvalues of periodic orbits of d-dimensional flows is immediate.

As stability of a flow can always be rewritten as stability of a Poincaré section return map, we find that the stability eigenvalues of any cycle, for a flow or a map in arbitrary dimension, is a metric invariant of the dynamical system.

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Commentary

Remark 6.1 Rectification of maps. The methods outlined above are standard in the analysis of fixed points and construction of normal forms for bifurcations, see for example ref. [1.11, 6.2, 6.4, 6.5, 6.6, 6.7, 6.8, 6.9, 3.5]. The geometry underlying such methods is pretty, and we enjoyed reading, for example, Percival and Richards [6.10], chaps. 2 and 4 of Ozorio de Almeida's monograph [6.11], and, as always, Arnol'd [6.1].

Recursive formulas for evaluation of derivatives needed to evaluate (6.11) are given, for example, in Appendix A of ref. [7.3]).

Résumé

Dynamics $(\mathcal{M}, f)$ is invariant under the group of all smooth conjugacies

$$(\mathcal{M}, f) \;\to\; (\mathcal{M}', g) = (h(\mathcal{M}), h \circ f \circ h^{-1}) \,.$$

This invariance can be used to (i) find a simplified representation for the flow and (ii) identify a set of invariants, numbers computed within a particular choice of $(\mathcal{M}, f)$, but invariant under all $\mathcal{M} \to h(\mathcal{M})$ smooth conjugacies.

The $2D$-dimensional phase space of an integrable Hamiltonian system of D degrees of freedom is fully foliated by D-tori. In the same spirit, for a uniformly hyperbolic, chaotic dynamical system one would like to change into a coordinate frame where the stable/unstable manifolds form a set of transversally interesecting hyper-planes, with the flow everywhere locally hyperbolic. That, cannot be done in general: Fully globally integrable and fully globally chaotic flows are a very small subset of all possible flows.

Nevertheless, we can profitably straighten out local neighborhoods of periodic orbits. Periodic orbits play a central role in any invariant characterization of the dynamics, as they form an infinite set of *metric invariants*: The stability eigenvalues of a periodic orbit remain invariant under any smooth nonlinear change of coordinates $f \to h \circ f \circ h^{-1}$.

What we *really* care about is developping invariant notions of what a given dynamical system is. The totality of smooth one-to-one nonlinear coordinate transformations h which map all trajectories of a given dynamical system $(\mathcal{M}, f^t)$ onto all trajectories of dynamical systems $(\mathcal{M}', g^t)$ gives us a huge equivalence class, much larger than the equivalence classes familiar from the theory of linear transformations, such as the rotation group $O(d)$ or the Galilean group of all rotations and translations in $\mathbb{R}^d$. In the theory of Lie groups, the full invariant specification of an object is given by a finite set of Casimir invariants. What a good full set of invariants for a group of general nonlinear smooth conjugacies might be is not known, but the set of all periodic orbits and their stability eigenvalues will turn out to be a good start.

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