Chapter 13

Why does it work?

> Bloch: "Space is the field of linear operators." Heisenberg: "Nonsense, space is blue and birds fly through it."
> Felix Bloch, *Heisenberg and the early days of quantum mechanics*

(R. Artuso, H.H. Rugh and P. Cvitanović)

The trace formulas and spectral determinants work well, sometimes very well indeed. The question is: why? The heuristic manipulations of chapter 11 were naive and reckless, as we are facing infinite-dimensional vector spaces and singular integral kernels.

In this chapter we outline some of the ingredients in the proofs that put the above trace and determinant formulas on solid mathematical footing. This requires taking a closer look at the Perron-Frobenius operator from a mathematical point of view, since up to now we have talked about eigenvalues without any reference to an underlying function space. In sect. 13.1 we show, by a simple example, that the spectrum is quite sensitive to the regularity properties of the functions considered, so what we referred to as *the set of eigenvalues* acquires a meaning only if the functional setting is properly tuned: this sets the stage for a discussion of analyticity properties mentioned in chapter 12. The program is enunciated in sect. 13.2, with the focus on expanding maps. In sect. 13.3 we concentrate on piecewise real-analytic maps acting on appropriate densities. For expanding and hyperbolic flows analyticity leads to a very strong result; not only do the determinants have better analyticity properties than the trace formulas, but the spectral determinants are singled out as being entire functions in the complex s plane.

This chapter is not meant to provide an exhaustive review of rigorous

results about properties of the Perron-Frobenius operator or analyticity results of spectral determinants or dynamical zeta functions (see remark 13.1), but rather to point out that heuristic considerations about traces and determinant can be put on firmer bases, under suitable hypotheses, and the mathematics behind this construction is both hard and profound.

If you are primarily interested in physical applications of periodic orbit theory, you should probably skip this chapter on the first reading.

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13.1 The simplest of spectral determinants: A single fixed point

In order to get some feeling for the determinants defined so formally in sect. 12.2, let us work out a trivial example: a repeller with only one expanding linear branch

$$f(x) = \Lambda x\,, \qquad |\Lambda| > 1\,,$$

and only one fixed point $x = 0$. The action of the Perron-Frobenius operator (7.10) is

$$\mathcal{L}\phi(y) = \int dx\, \delta(y - \Lambda x)\, \phi(x) = \frac{1}{|\Lambda|}\phi(y/\Lambda)\,. \tag{13.1}$$

From this one immediately gets that the monomials y^n are eigenfunctions:

$$\mathcal{L}y^n = \frac{1}{|\Lambda|\Lambda^n} y^n\,, \qquad n = 0, 1, 2, \ldots \tag{13.2}$$

We note that the eigenvalues Λ^{-n-1} fall off exponentially with n, and that the trace of $\mathcal{L}$ is

$$\mathrm{tr}\,\mathcal{L} = \frac{1}{|\Lambda|}\sum_{n=0}^{\infty}\Lambda^{-n} = \frac{1}{|\Lambda|(1-\Lambda^{-1})} = \frac{1}{|f(0)' - 1|}\,,$$

in agreement with (11.6). A similar result is easily obtained for powers of $\mathcal{L}$, and for the spectral determinant (12.3) one obtains:

$$\det(1-z\mathcal{L}) = \prod_{k=0}^{\infty}\left(1-\frac{z}{|\Lambda|\Lambda^k}\right) = \sum_{k=0}^{\infty} Q_k t^k\,, \qquad t=-z/|\Lambda|\,, \tag{13.3}$$

where the coefficients Q_k are given explicitly by the *Euler formula*

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$$Q_k = \frac{1}{1-\Lambda^{-1}}\,\frac{\Lambda^{-1}}{1-\Lambda^{-2}}\,\cdots\,\frac{\Lambda^{-k+1}}{1-\Lambda^{-k}}\quad . \tag{13.4}$$

(if you cannot figure out exercise 13.3 check the solutions on 771 for proofs of this formula).

Note that the coefficients Q_k decay asymptotically *faster* than exponentially, as $\Lambda^{-k(k-1)/2}$. As we shall see in sect. 13.3.1, these results carry over to any single-branch repeller. This super-exponential decay of Q_k ensures that for a repeller consisting of a single repelling point the spectral determinant (13.3) is *entire* in the complex z plane.

What is the meaning of (13.3)? It gives us an interpretation of the index k in the Selberg product representation of the spectral determinant (12.9): k labels the kth local fixed-point eigenvalue $1/|\Lambda|\Lambda^k$.

Now if the spectral determinant is entire, on the basis of (12.25) we get that the dynamical zeta function is a meromorphic function. These mathematical properties are of direct physical import: they guarantee that finite order estimates of zeroes of dynamical zeta functions and spectral determinants converge exponentially or super-exponentially to the exact values, and so the cycle expansions of chapter 15 represent a *true perturbative* approach to chaotic dynamics. To see how exponential convergence comes out of analytic properties we take the simplest possible model of a meromorphic function. Consider the function

$$h(z) \;=\; \frac{z-a}{z-b}$$

with a, b real and positive and $a < b$. Within the cycle $|z| < b$ we may represent h as a power series

$$h(z) \;=\; \sum_{k=0}^{\infty} \sigma_k z^k$$

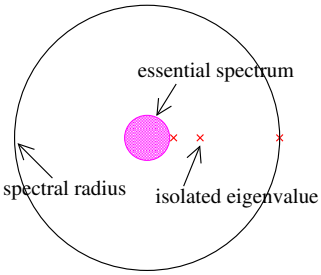

Figure 13.1: Spectrum for Perron-Frobenius operator in an extended function space: only a few isolated eigenvalues remain between the spectral radius and the essential spectral radius, bounding continuous spectrum

where $\sigma_0 = a/b$ and higher order coefficients are given by $\sigma_j = (a-b)/b^{j+1}$ Now we take the truncation of order N of the power series

$$h_N(z) = \sum_{k=0}^{N} \sigma_k z^k = \frac{a}{b} + \frac{z(a-b)(1-z^N/b^N)}{b^2(1-z/b)}\,.$$

Let $\hat{z}_N$ be the solution of the truncated series $h_N(\hat{z}_N) = 0$. To estimate the distance between a and $\hat{z}_N$ it is sufficient to calculate $h_N(a)$, which is of order $(a/b)^{N+1}$, and so finite order estimates indeed converge exponentially to the asymptotic value.

The discussion of our simple example confirms that our formal manipulations with traces and determinants are justified, namely the Perron-Frobenius operator has isolated eigenvalues: trace formulas are then explicitly verified, the spectral determinant is an analytic function whose zeroes yield the eigenvalues. Life is actually harder, as we may appreciate through the following considerations

- Our discussion tacitly assumed something that is physically entirely reasonable: our evolution operator is acting on the space of analytic functions, that is, we are allowed to represent the initial density $\rho(x)$ by its Taylor expansions in the neighborhoods of periodic points. This is however far from being the only possible choice: we might choose the function space $C^{k+\alpha}$, that is the space of k times differentiable functions whose k'th derivatives are Hölder continuous with an exponent $0 < \alpha \leq 1$: then every y^η with $\mathrm{Re}\,\eta > k$ is an eigenfunction of Perron-Frobenius operator and we have

  $$\mathcal{L} y^\eta = \frac{1}{|\Lambda|\Lambda^\eta} y^\eta$$

  This spectrum is quite different from the analytic case: only a small number of isolated eigenvalues remain, enclosed between the unit disk

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and a smaller disk of radius $1/|\Lambda|^{k+1}$, (the so-called essential spectral radius) see fig. 13.1.

In sect. 13.2 we will discuss this point further, with the aid of a less trivial one-dimensional example. We remark that our point of view is complementary to the standard setting of ergodic theory, where many chaotic properties of a dynamical system are encoded by the presence of a *continuous* spectrum, which is necessary in order to prove asymptotic decay of correlations in $L^2(d\mu)$ setting.

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- A deceptively innocent assumption hides behind many features discussed so far: that (13.1) maps a given function space into itself. This is strictly related to the *expanding* property of the map: if $f(x)$ is smooth in a domain D then $f(x/\Lambda)$ is smooth on a *larger* domain, provided $|\Lambda| > 1$. This is not obviously the case for hyperbolic systems in higher dimensions, and, as we shall see in sect. 13.3, extensions of the results obtained for expanding maps will be highly nontrivial,

- It is not a priori clear that the above analysis of a simple one-branch, one fixed point repeller can be extended to dynamical systems with a Cantor set infinity of periodic points: we show that next.

13.2 Analyticity of spectral determinants

> They savored the strange warm glow of being much more ignorant than ordinary people, who were only ignorant of ordinary things.
> Terry Pratchett

We now choose another paradigmatic example (the Bernoulli shift) and sketch the steps that lead to the proof that the corresponding spectral determinant is an entire function. Before doing that it is convenient to summarize a few facts about classical theory of integral equations.

13.2.1 Classical Fredholm theory

> He who would valiant be
> 'Gainst all disaster
> Let him in constancy
> Follow the Master.
> John Bunyan, *Pilgrim's Progress*

The Perron-Frobenius operator

$$\mathcal{L}\phi(x) = \int dy\, \delta(x - f(y))\, \phi(y)$$

has the same appearance as a classical Fredholm integral operator

$$\mathcal{K}\varphi(x) \;=\; \int_{\mathcal{Q}} dy\, \mathcal{K}(x,y)\varphi(y)\,, \tag{13.5}$$

and one is tempted to resort to the classical Fredholm theory in order to establish analyticity properties of spectral determinants. This path to enlightenment is blocked by the singular nature of the kernel, which is a distribution, whereas the standard theory of integral equations usually concerns itself with regular kernels $\mathcal{K}(x,y) \in L^2(\mathcal{Q}^2)$. Here we briefly recall some steps of the Fredholm theory, before going to our major example in sect. 13.2.2.

The general form of Fredholm integral equations of the second kind is

$$\varphi(x) \;=\; \int_{\mathcal{Q}} dy\, \mathcal{K}(x,y)\varphi(y) \;+\; \xi(x) \tag{13.6}$$

where $\xi(x)$ is a given function in $L^2(\mathcal{Q})$ and the kernel $\mathcal{K}(x,y) \in L^2(\mathcal{Q}^2)$ (Hilbert-Schmidt condition). The natural object to study is then the linear integral operator (13.5), acting on the Hilbert space $L^2(\mathcal{Q})$: and the fundamental property that follows from the $L^2(Q)$ nature of the kernel is that such an operator is *compact*, that is close to a finite rank operator (see appendix K). A compact operator has the property that for every $\delta > 0$ only a *finite* number of linearly independent eigenvectors exist corresponding to eigenvalues whose absolute value exceeds δ, so we immediately realize (fig. 13.1) that much work is needed to bring Perron-Frobenius operators into this picture.

We rewrite (13.6) in the form

$$\mathcal{T}\varphi \;=\; \xi\,, \mathcal{T} \;=\; \mathbb{1} - \mathcal{K}\,. \tag{13.7}$$

The Fredholm alternative is now stated as follows: the equation $\mathcal{T}\varphi = \xi$ as a unique solution for every $\xi \in L^2(\mathcal{Q})$ **or** there exists a non-zero solution of $\mathcal{T}\varphi_0 = 0$, with an eigenvector of $\mathcal{K}$ corresponding to the eigenvalue 1.

The theory remains the same if instead of $\mathcal{T}$ we consider the operator $\mathcal{T}_\lambda = \mathbb{1} - \lambda\mathcal{K}$ with $\lambda \neq 0$. As $\mathcal{K}$ is a compact operator there will be at most a denumerable set of λ for which the second part of Fredholm alternative holds: so apart from this set the inverse operator $(\mathbb{1} - \lambda\mathcal{T})^{-1}$ exists and is a bounded operator. When λ is sufficiently small we may look for a perturbative expression for such an inverse, as a geometric series

$$(\mathbb{1} - \lambda\mathcal{K})^{-1} = \mathbb{1} + \lambda\mathcal{K} + \lambda^2\mathcal{K}^2 + \cdots = \mathbb{1} + \lambda\mathcal{W}\,, \tag{13.8}$$

where each $\mathcal{K}^n$ is still a compact integral operator with kernel

$$\mathcal{K}^n(x,y) = \int_{\mathcal{Q}^{n-1}} dz_1 \ldots dz_{n-1}\, \mathcal{K}(x,z_1)\cdots\mathcal{K}(z_{n-1},y)\,,$$

and $\mathcal{W}$ is also compact, as it is given by the convergent sum of compact operators. The problem with (13.8) is that the series has a finite radius of convergence, while apart from a denumerable set of λ's the inverse operator is well defined. A fundamental result in the theory of integral equations consists in rewriting the resolving kernel $\mathcal{W}$ as a ratio of two *analytic* functions of λ

$$\mathcal{W}(x,y) = \frac{\mathcal{D}(x,y;\lambda)}{D(\lambda)}.$$

If we introduce the notation

$$\mathcal{K}\begin{pmatrix} x_1 \ldots x_n \\ y_1 \ldots y_n \end{pmatrix} = \begin{vmatrix} \mathcal{K}(x_1,y_1) & \ldots & \mathcal{K}(x_1,y_n) \\ \ldots & \ldots & \ldots \\ \mathcal{K}(x_n,y_1) & \ldots & \mathcal{K}(x_n,y_n) \end{vmatrix}$$

we may write the explicit expressions

$$\begin{aligned} D(\lambda) &= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n}{n!} \int_{\mathcal{Q}^n} dz_1 \ldots dz_n\, \mathcal{K}\begin{pmatrix} z_1 \ldots z_n \\ z_1 \ldots z_n \end{pmatrix} \\ &= \exp - \sum_{m=1}^{\infty} \frac{\lambda^m}{m} \operatorname{tr} \mathcal{K}^m \end{aligned} \tag{13.9}$$

and

$$\mathcal{D}(x,y;\lambda) = \mathcal{K}\begin{pmatrix} x \\ y \end{pmatrix} + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_{\mathcal{Q}^n} dz_1 \ldots dz_n\, \mathcal{K}\begin{pmatrix} x & z_1 & \ldots & z_n \\ y & z_1 & \ldots & z_n \end{pmatrix}$$

$D(\lambda)$ is known as the Fredholm determinant (see (12.24) and appendix K): it is an entire analytic function of λ, and $D(\lambda) = 0$ only if $1/\lambda$ is an eigenvalue of $\mathcal{K}$.

We remark again that the whole theory is based on the compactness of the integral operator, that is on the functional properties (summability) of its kernel.

13.2.2 Bernoulli shift

Consider now the Bernoulli shift

$$x \mapsto 2x \bmod 1 \qquad\qquad x \in [0,1] \tag{13.10}$$

and look at spectral properties in appropriate function spaces. The Perron-Frobenius operator associated with this map is given by

$$\mathcal{L}h(y) = \frac{1}{2}h\left(\frac{y}{2}\right) + \frac{1}{2}h\left(\frac{y+1}{2}\right) . \tag{13.11}$$

Spaces of summable functions as $L^1([0,1])$ or $L^2([0,1])$ are mapped into themselves by the Perron-Frobenius operator, and in both spaces the constant function $h \equiv 1$ is an eigenfunction with eigenvalue 1. This obviously does not exhaust the spectrum: if we focus our attention on $L^1([0,1])$ we also have a whole family of eigenfunctions, parametrized by complex θ with $\mathrm{Re}\,\theta > 0$. One verifies that

$$h_\theta(y) = \sum_{k\neq 0} \exp(2\pi i k y)\frac{1}{|k|^\theta} \tag{13.12}$$

is indeed an L^1-eigenfunction with (complex) eigenvalue $2^{-\theta}$, by varying θ one realizes that such eigenvalues fill out the entire unit disk. This casts out a 'spectral rug', also known as an essential spectrum, which hides all the finer details of the spectrum.

For a bounded linear operator $\mathcal{A}$ on a Banach space $\mathbf{\Omega}$, the spectral radius is the smallest positive number ρ_{spec} such the spectrum is inside the disk of radius ρ_{spec}, while the essential spectral radius is the smallest positive number ρ_{ess} such that outside the disk of radius ρ_{ess} the spectrum consists only of isolated eigenvalues of finite multiplicity (see fig. 13.1).

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We may shrink the essential spectrum by letting the Perron-Frobenius operator act on a space of *smoother* functions, exactly as in the one-branch

repeller case of sect. 13.1. We thus consider a smaller space, $C^{k+\alpha}$, the space of k times differentiable functions whose k'th derivatives are Hölder continuous with an exponent $0 < \alpha \leq 1$: the expansion property guarantees that such a space is mapped into itself by the Perron-Frobenius operator. In the strip $0 < \mathrm{Re}\ \theta < k + \alpha$ most h_θ will cease to be eigenfunctions in the space $C^{k+\alpha}$. Only for integer valued $\theta = n$ the function h_n survives. In this way we arrive at a finite set of *isolated* eigenvalues 1, 2^{-1}, $\cdots$, 2^{-k}, and an essential spectral radius $\rho_{ess} = 2^{-(k+\alpha)}$.

For this simple example, we may actually exactly write down the eigenfunctions: they coincide, up to a constant, with the Bernoulli polynomials $B_n(x)$. These polynomials are defined as successive derivatives of $te^{xt}/(e^t - 1)$ evaluated at $t = 0$:

$$\mathcal{G}_t(x) \ = \frac{te^{xt}}{e^t - 1} \ = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

so $B_0(x) = 1$, $B_1(x) = x - 1/2$, etc. .

If we let the Perron-Frobenius operator (13.11) act on the generating function $\mathcal{G}$, we get

$$\mathcal{L}\mathcal{G}_t(x) \ = \ \frac{1}{2}\left(\frac{te^{xt/2}}{e^t - 1} + \frac{te^{t/2}e^{xt/2}}{e^t - 1} \right) \ = \ \frac{t/2e^{xt/2}}{e^{t/2} - 1} \ = \sum_{n=1}^{\infty} B_n(x) \frac{(t/2)^n}{n!}$$

it follows that each $B_n(x)$ is an eigenfunction of the Perron-Frobenius operator $\mathcal{L}$ with eigenvalue $1/2^n$. The persistence of a finite essential spectral radius would suggest that traces and determinants do not exist in this case either. The pleasant surprise is that they do, see remark 13.3.

We follow a simpler path and restrict the function space even further, namely to a space of analytic functions, i.e. for which the is convergent at each point of the interval $[0, 1]$. With this choice things turn out easy and elegant. To be more specific let h be a holomorphic and bounded function on the disk $D = B(0, R)$ of radius $R > 0$ centered at the origin. Our Perron-Frobenius operator preserves the space of such functions provided $(1 + R)/2 < R$ so all we need is to choose $R > 1$. In this the expansion property of the Bernoulli shift enter). If F denotes one of the inverse branches of the Bernoulli shift (13.10) the corresponding part of the Perron-Frobenius operator is given by $\mathcal{L}_F h(y) = s\ F'(y)\ h \circ F(y)$, using the Cauchy integral formula:

$$\mathcal{L}_F h(y) = s \oint_{\partial D} \frac{h(w)F'(y)}{w - F(y)} dw.$$

For reasons that will be made clear later we have introduced a sign $s = \pm 1$ of the given real branch $|F'(y)| = sF(y)$. For both branches of the Bernoulli shift $s^2 + 1$, one is not allowed to take absolute values as this could destroy analyticity. In the above formula one may also replace the domain D by *any domain* containing $[0, 1]$ such that the inverse branches maps the closure of D into the interior of D. Why? simply because the kernel stays non-singular under this condition, i.e. $w - F(y) \neq 0$ whenever $w \in \partial D$ and $y \in \mathrm{Cl}\, D$.

The problem is now reduced to the standard theory for Fredholm determinants. The integral kernel is no longer singular, traces and determinants are well-defined and we may even calculate the trace of $\mathcal{L}_F$ as a contour integral:

$$\mathrm{tr}\ \mathcal{L}_F = \oint \frac{sF'(w)}{w - F(w)} dw.$$

Elementary complex analysis shows that since F maps the closure of D into its own interior, F has a unique (real-valued) fixed point x^* with a multiplier strictly smaller than one in absolute value. Residue calculus therefore yields

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$$\mathrm{tr}\ \mathcal{L}_F = \frac{sF'(x^*)}{1 - F'(x^*)} = \frac{1}{|f'(x^*) - 1|},$$

justifies our previous ad hoc calculations of traces by means of Dirac delta functions. The full operator has two components corresponding to the two branches og the . For the n times iterated operator we have a full binary shift and for each of the 2^n branches the above calculations carry over in each , yielding the trace $(2^n - 1)^{-1}$. Without further ado we substitute everything back and obtain the determinant,

$$\det(1 - z\mathcal{L}) = \exp\left(- \sum_{n=1} \frac{z^n}{n} \frac{2^n}{2^n - 1} \right) = \prod_{k=0} \left(1 - \frac{z}{2^k}\right),$$

verifying the fact that the Bernoulli polynomials are eigenfunctions with eigenvalues $1/2^n$, $n = 0, 1, 2, \ldots$.

We worked out a very specific example, yet our conclusions can be generalized, provided a number of restrictive requirements are met by our dynamical systems:

1) the evolution operator is *multiplicative* along the flow,
2) the symbolic dynamics is a *finite subshift*,
3) all cycle eigenvalues are *hyperbolic* (exponentially bounded away from 1),
4) the map (or the flow) is *real analytic*, that is it has a piecewise analytic continuation to a complex extension of the phase space.

These assumptions are romantic projections not lived up to by the dynamical systems that we actually desire to understand. Still, they are not devoid of physical interest; for example, nice repellers like our 3-disk game of pinball of changes do satisfy the above requirements.

Properties 1 and 2 enable us to represent the evolution operator as a matrix in an appropriate basis space; properties 3 and 4 enable us to bound the size of the matrix elements and control the eigenvalues. To see what can go wrong consider the following examples:

Property 1 is violated for flows in 3 or more dimensions by the following weighted evolution operator

$$\mathcal{L}^t(y,x) = |\Lambda^t(x)|^\beta \delta\big(y - f^t(x)\big) \ ,$$

where $\Lambda^t(x)$ is an eigenvalue of the Jacobian matrix transverse to the flow. Semiclassical quantum mechanics suggest operators of this form with $\beta = 1/2$, (see chapter 27). The problem with such operators is due to the fact that when considering the Jacobian matrices $\mathbf{J}_{ab} = \mathbf{J}_a\mathbf{J}_b$ for two successive trajectory segments a and b, the corresponding eigenvalues are in general *not* multiplicative, $\Lambda_{ab} \neq \Lambda_a\Lambda_b$ (unless a, b are repeats of the same prime cycle p, so $\mathbf{J}_a\mathbf{J}_b = \mathbf{J}_p^{r_a+r_b}$). Consequently, this evolution operator is not multiplicative along the trajectory. The theorems require that the evolution be represented as a matrix in an appropriate polynomial basis, and thus cannot be applied to non-multiplicative kernels, that is kernels that do not satisfy the semi-group property $\mathcal{L}^{t'} \circ \mathcal{L}^t = \mathcal{L}^{t'+t}$. Cure for this problem in this particular case will be given in sect. H.1.

Property 2 is violated by the 1-d tent map (see fig. 13.2)

$$f(x) = \alpha(1 \ - \ |1-2x|) \ , \quad 1/2 < \alpha < 1 \ .$$

All cycle eigenvalues are hyperbolic, but in general the critical point $x_c = 1/2$ is not a pre-periodic point, there is no finite Markov partition and the symbolic dynamics does not have a finite grammar. In practice this means that while the leading eigenvalue of $\mathcal{L}$ might be computable, the rest of the

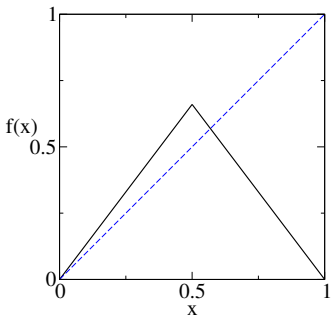

Figure 13.2: A (hyperbolic) tent map without a finite Markov partition.

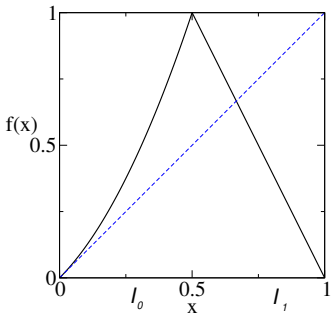

Figure 13.3: A Markov map with a marginal fixed point.

spectrum is very hard to control; as the parameter α is varied, non-leading zeros of the spectral determinant move wildly about.

Property 3 is violated by the map (see fig. 13.3)

$$f(x) = \begin{cases} x + 2x^2 & , \quad x \in I_0 = [0, \frac{1}{2}] \\ 2 - 2x & , \quad x \in I_1 = [\frac{1}{2}, 1] \end{cases} .$$

Here the interval $[0, 1]$ has a Markov partition into the two subintervals I_0 and I_1; f is monotone on each. However, the fixed point at $x = 0$ has marginal stability $\Lambda_0 = 1$, and violates the condition 3. This type of map is called intermittent and necessitates much extra work. The problem is that the dynamics in the neighborhood of a marginal fixed point is very slow, with correlations decaying as power laws rather than exponentially. We will discuss such flows in chapter 18.

The property 4 is required as the heuristic approach of chapter 11 faces two major hurdles:

1. The trace (11.7) is not well defined since the integral kernel is singular.
2. The existence and properties of eigenvalues are by no means clear.

Actually this property is quite restrictive, but we need it in the present approach, in order that the Banach space of analytic functions in a disk is preserved by the Perron-Frobenius operator.

In attempting to generalize the results we encounter several problems. First, in higher dimensions life is not as simple. Multi-dimensional residue calculus is at our disposal but in general requires that we find poly-domains (direct product of domains in each coordinate) and this need not be the case. Second, and perhaps somewhat surprisingly, the 'counting of periodic orbits' presents a difficult problem. For example, instead of the Bernoulli shift consider the doubling map of the circle, $x \mapsto 2x \mod 1$, $x \in R/Z$. Compared to the shift on the interval $[0,1]$ the only difference is that the endpoints 0 and 1 are now glued together. But since these endpoints are fixed points of the map the number of cycles of length n decreases by 1. The determinant becomes:

$$\det(1 - z\mathcal{L}) = \exp\left(-\sum_{n=1} \frac{z^n}{n} \frac{2^n - 1}{2^n - 1}\right) = 1 - z. \tag{13.13}$$

The value $z = 1$ still comes from the constant eigenfunction but the Bernoulli polynomials no longer contribute to the spectrum (they are not periodic). Proofs of these facts, however, are difficult if one sticks to the space of analytic functions.

Third, our Cauchy formulas *a priori* work only when considering purely expanding maps. When stable and unstable directions co-exist we have to resort to stranger function spaces, as shown in the next section.

13.3 Hyperbolic maps

(H.H. Rugh)

Moving on to hyperbolic systems, one faces the following paradox: If f is an area-preserving hyperbolic and real-analytic map of e.g. a two dimensional torus then the Perron-Frobenius operator is clearly unitary on the space of L^2 functions. The spectrum is then confined to the unit-circle. On the other hand when we compute determinants we find eigenvalues scattered around inside the unit disk. Thinking back on our Bernoulli shift example one would like to imagine these eigenvalues as popping up from the L^2 spectrum by shrinking the function space. Shrinking the space, however, can only make the spectrum smaller so this is obviously not what happens. Instead one needs to introduce a 'mixed' function space where in the unstable direction one resort to analytic functions as before but in the stable

direction one considers a 'dual space' of distributions on analytic functions. Such a space is neither included in nor does it include the L^2-space and we have thus resolved the paradox. But it still remains to be seen how traces and determinants are calculated.

First, let us consider the apparently trivial linear example ($0 < \lambda_s < 1$, $\Lambda_u > 1$):

$$f(z) = (f_1(z_1, z_2), f_2(z_1, z_2)) = (\lambda_s z_1, \Lambda_u z_2) \tag{13.14}$$

The function space, alluded to above, is then a mixture of Laurent series in the z_1 variable and analytic functions in the z_2 variable. Thus, one considers expansions in terms of $\varphi_{n_1,n_2}(z_1, z_2) = z_1^{-n_1-1} z_2^{n_2}$ with $n_1, n_2 = 0, 1, 2, \ldots$ If one looks at the corresponding Perron-Frobenius operator, one gets a simple generalization of the 1-d repeller:

$$\mathcal{L}h(z_1, z_2) = \frac{1}{\lambda_s \cdot \Lambda_u} h(z_1/\lambda_s, z_2/\Lambda_u) \tag{13.15}$$

The action of Perron-Frobenius operator on the basis functions yields

$$\mathcal{L}\varphi_{n_1,n_2}(z_1, z_2) \;=\; \frac{\lambda_s^{n_1}}{\Lambda_u^{1+n_2}} \varphi_{n_1,n_2}(z_1, z_2)$$

so that the above basis elements are eigenvectors with eigenvalues $\lambda_s^{n_1} \Lambda_u^{-n_2-1}$ and one verifies by an explicit calculation that the trace indeed equals $\det(f' - \mathbf{1})^{-1} = (\Lambda_u - 1)^{-1}(1 - \lambda_s)^{-1}$.

This example is somewhat misleading, however, as we have made explicit use of an analytic 'splitting' into stable/unstable directions. For a more general hyperbolic map, if one attempts to make such a splitting it will not be analytic and the whole argument falls apart. Nevertheless, one may introduce 'almost' analytic splittings and write down a generalization of the above operator as follows (s is the signature of the derivative in the unstable direction):

$$\mathcal{L}h(z_1, z_2) = \oint \oint \frac{s\; h(w_1, w_2)}{(z_1 - f_1(w_1, w_2)(f_2(w_1, w_2) - z_2)} \frac{dw_1}{2\pi i} \frac{dw_2}{2\pi i}. \tag{13.16}$$

Here the 'function' h should belong to a space of functions analytic respectively *outside* a disk and *inside* a disk in the first and the second coordinate and with the additional property that the function decays to zero as the first

coordinate tends to infinity. The contour integrals are along the boundaries of these disks. It is but an exercise in multi-dimensional residue calculus to verify that for the above linear example this expression reduces to (13.15). Such operators form the building bricks in the calculation of traces and determinants and one is able to prove the following:

Theorem: *The spectral determinant for hyperbolic analytic maps is entire.*

The proof, apart from the Markov property which is the same as for the purely expanding case, relies heavily on analyticity of the map in the explicit construction of the function space. As we have also seen in the previous example the basic idea is to view the hyperbolicity as a cross product of a contracting map in the forward time and another contracting map in the backward time. In this case the Markov property introduced above has to be elaborated a bit. Instead of dividing the phase space into intervals, one divides it into rectangles. The rectangles should be viewed as a direct product of intervals (say horizontal and vertical), such that the forward map is contracting in, for example, the horizontal direction, while the inverse map is contracting in the vertical direction. For Axiom A systems (see remark 13.11) one may choose coordinate axes close to the stable/unstable manifolds of the map. With the phase space divided into N rectangles $\{\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_N\}$, $\mathcal{M}_i = I_i^h \times I_i^v$ one needs complex extension $D_i^h \times D_i^v$, with which the hyperbolicity condition (which at the same time guarantees the Markov property) can be formulated as follows:

Analytic hyperbolic property: Either $f(\mathcal{M}_i) \cap \mathrm{Int}(\mathcal{M}_j) = \emptyset$, or for each pair $w_h \in \mathrm{Cl}(D_i^h)$, $z_v \in \mathrm{Cl}(D_j^v)$ there exist unique analytic functions of w_h, z_v: $w_v = w_v(w_h, z_v) \in \mathrm{Int}(D_i^v)$, $z_h = z_h(w_h, z_v) \in \mathrm{Int}(D_j^h)$, such that $f(w_h, w_v) = (z_h, z_v)$. Furthermore, if $w_h \in I_i^h$ and $z_v \in I_j^v$, then $w_v \in I_i^v$ and $z_h \in I_j^h$ (see fig. 13.4).

What this means for the iterated map is that one replaces coordinates z_h, z_v at time n by the contracting pair z_h, w_v, where w_v is the contracting coordinate at time $n+1$ for the 'partial' inverse map.

In two dimensions the operator in (13.16) is acting on functions analytic outside D_i^h in the horizontal direction (and tending to zero at infinity) and inside D_i^v in the vertical direction. The contour integrals are precisely along the boundaries of these domains.

A map f satisfying the above condition is called analytic hyperbolic

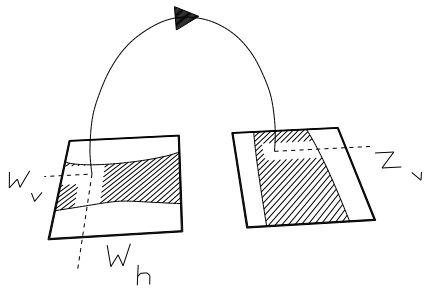

Figure 13.4: For an analytic hyperbolic map, specifying the contracting coordinate w_h at the initial rectangle and the expanding coordinate z_v at the image rectangle defines a unique trajectory between the two rectangles. In particular, w_v and z_h (not shown) are uniquely specified.

and the theorem states that the associated spectral determinant is entire, and that the trace formula (11.7) is correct.

13.3.1 Matrix representations

When considering analytic maps there is another, and for numerical purposes, sometimes convenient way to look at the operators, namely through matrix representations. The size of these matrices is infinite but entries in the matrix decay exponentially fast with the indisize. Hence, within an exponentially small error one may safely do calculations using finite matrix truncations.

Furthermore, from bounds on the elements L_{mn} one calculates bounds on tr $\left(\wedge^k \mathcal{L}\right)$ and verifies that they fall off as $\Lambda^{-k^2/2}$, concluding that the $\mathcal{L}$ eigenvalues fall off exponentially for a general Axiom A 1-d map. In order to illustrate how this works, we work out a simple example.

As in sect. 13.1 we start with a map with a single fixed point, but this time with a nonlinear map f with a nonlinear inverse $F = f^{-1}$, $s = \mathrm{sgn}(F')$

$$\mathcal{L} \circ \phi(z) = \int dx\, \delta(z - f(x))\, \phi(x) = s\, F'(z)\, \phi(F(z))\,.$$

Assume that F is a contraction of the unit disk, that is

$$|F(z)| < \theta < 1 \quad \text{and} \quad |F'(z)| < C < \infty \quad \text{for} \quad |z| < 1\,, \tag{13.17}$$

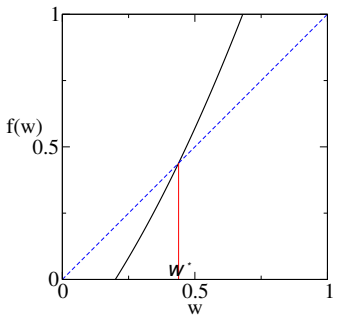

Figure 13.5: A nonlinear one-branch repeller with a single fixed point w^*.

and expand ϕ in a polynomial basis by means of the Cauchy formula

$$\phi(z) = \sum_{n\geq 0} z^n \phi_n = \oint \frac{dw}{2\pi i} \frac{\phi(w)}{w-z}, \quad \phi_n = \oint \frac{dw}{2\pi i} \frac{\phi(w)}{w^{n+1}}$$

In this basis, $\mathcal{L}$ is a represented by the matrix

$$\mathcal{L} \circ \phi(w) = \sum_{m,n} w^m L_{mn} \phi_n \,, \quad L_{mn} = \oint \frac{dw}{2\pi i} \frac{s\ F'(w)(F(w))^n}{w^{m+1}} \,. \tag{13.18}$$

Taking the trace and summing we get:

$$\text{tr}\ \mathcal{L} = \sum_{n\geq 0} L_{nn} = \oint \frac{dw}{2\pi i} \frac{s\ F'(w)}{w - F(w)} \,.$$

This integral has but one simple pole at the unique fix point $w^* = F(w^*) = f(w^*)$. Hence

$$\text{tr}\ \mathcal{L} = \frac{s\ F'(w^*)}{1 - F'(w^*)} = \frac{1}{|f'(w^*) - 1|} \,.$$

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We recognize this result as a generalization of the single piecewise-linear fixed-point example (13.2), $\phi_n = y^n$, and $\mathcal{L}$ is diagonal (no sum on repeated n here), $L_{nn} = 1/|\Lambda|\Lambda^{-n}$, so we have verified the heuristic trace formula for an expanding map with a single fixed point. The requirement that map be analytic is needed to substitute bound (13.17) into the contour integral (13.18) and obtain the inequality

$$|L_{mn}| \leq \sup_{|w|\leq 1} |F'(w)|\ |F(w)|^n \leq C\theta^n$$

which shows that finite $[N \times N]$ matrix truncations approximate the operator within an error exponentially small in N. It also follows that eigenvalues fall off as θ^n. In higher dimension similar considerations show that the entries in the matrix fall off as $1/\Lambda^{k^{1+1/d}}$, and eigenvalues as $1/\Lambda^{k^{1/d}}$.

13.4 Physics of eigenvalues and eigenfunctions

We appreciate by now that any serious attempt to look at spectral properties of the Perron-Frobenius operator involves hard mathematics: but the effort is rewarded by the fact that we are finally able to control analyticity properties of dynamical zeta functions and spectral determinants, and thus substantiate the claim that these objects provide a powerful and well founded perturbation theory.

Quite often (see for instance chapter 8) the physical interest is concentrated in the leading eigenvalue, as it gives the escape rate from a repeller, or, when considering generalized transfer operators, it yields expressions for generating functions for observables. We recall (see chapter 7) that also the eigenfunction associated to the leading eigenvalue has a remarkable property: it provides the density of the invariant measure, with singular measures ruled out by the choice of the function space. Such a conclusion is coherent with a the validity of a generalized Perron-Frobenius theorem for the evolution operator. In the finite dimensional setting such theorem is formulated as follows:

- let L_{nm} be a nonnegative matrix, such that some n exists for which $(L^n)_{ij} > 0 \ \forall i, j$: then
    1. the maximal modulus eigenvalue is non degenerate, real and positive
    2. the corresponding eigenvector (defined up to a constant) has nonnegative coordinates

We may ask what physical information is contained in eigenvalues beyond the leading one: suppose that we have a probability conserving system (so that the dominant eigenvalue is 1), for which the essential spectral radius is such that $0 < \rho_{ess} < \theta < 1$ on some Banach space $\mathcal{B}$ and denote by $\mathbf{P}$ the projection corresponding to the part of the spectrum inside a disk of radius θ. We denote by λ_1, $\lambda_2 \ldots \lambda_M$ the eigenvalues outside of this disk, ordered by the size of their absolute value (so that $\lambda_1 = 1$). Then we have

the following decomposition

$$\mathcal{L}\varphi \;=\; \sum_{i=1}^{M} \lambda_i \psi_i L_i \psi_i^* \varphi \;+\; \mathbf{P}\mathcal{L}\varphi \tag{13.19}$$

when L_i are (finite) matrices in Jordan normal form ($L_1 = 1$ is a 1×1 matrix, as λ_1 is simple, due to Perron-Frobenius theorem), while ψ_i is a row vector whose elements are a basis on the eigenspace corresponding to λ_i, and ψ_i^* is a column vector of elements of $\mathcal{B}^*$ (the dual space, of linear functionals over $\mathcal{B}$) spanning the eigenspace of $\mathcal{L}^*$ corresponding to λ_i. For iterates of Perron-Frobenius operator (13.19) becomes

$$\mathcal{L}^n\varphi \;=\; \sum_{i=1}^{M} \lambda_i^n \psi_i L_i^n \psi_i^* \varphi \;+\; \mathbf{P}\mathcal{L}^n\varphi \tag{13.20}$$

If we now consider expressions like

$$C(n)_{\xi,\varphi} = \int_{\mathcal{M}} dy\, \xi(y)\, (\mathcal{L}^n\varphi)\,(y) \;=\; \int_{\mathcal{M}} dw\, (\xi \circ f^n)(w)\varphi(w) \tag{13.21}$$

we have

$$C(n)_{\xi,\varphi} \;=\; \lambda_1^n \omega_1(\xi,\varphi) + \sum_{i=2}^{L} \lambda_i^n \omega(n)_i(\xi,\varphi) + \mathcal{O}(\theta^n) \tag{13.22}$$

where

$$\omega(n)_i(\xi,\varphi) \;=\; \int_{\mathcal{M}} dy\, \xi(y) \psi_i L_i^n \psi_i^* \varphi$$

In this way we see how eigenvalues beyond the leading one provide a twofold piece of information: they rule the convergence of expressions containing high powers of evolution operator to the leading order (the λ_1 contribution). Moreover if $\omega_1(\xi,\varphi) = 0$ then (13.21) defines a correlation function: as each term in (13.22) vanishes exponentially in the $n \to \infty$ limit, the eigenvalues $\lambda_2, \ldots \lambda_M$ rule the exponential decay of correlations for our dynamical system. We observe that prefactors ω depend on the choice of functions, while the exponential decay rates (logarithms of λ_i) do not: the correlation spectrum is thus an *universal* property of the dynamics (once we fix the overall functional space our Perron-Frobenius operator acts on).

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So let us come back the Bernoulli shift example (13.10), on the space of analytic functions on a disk: apart from the origin we have only simple eigenvalues $\lambda_k = 2^{-k}$ $k = 0, 1, \ldots$. The eigenvalue $\lambda_0 = 1$ corresponds to probability conservation: the corresponding eigenfunction $B_0(x) = 1$ indicates that the natural, measure has a constant density over the unit interval. If we now take any analytic function $\eta(x)$ with zero average (with respect to the Lebesgue measure), we have that $\omega_1(\eta, \eta) = 0$, and from (13.22) we have that the asymptotic decay of correlation function is (unless also $\omega_1(\eta, \eta) = 0$)

$$C_{\eta,\eta}(n) \sim \exp(-n \log 2) \tag{13.23}$$

thus $-\log \lambda_1$ gives the exponential decay rate of correlations (with a prefactor that depends on the choice of the function). Actually the Bernoulli shift case may be treated exactly, as for analytic functions we can employ the Euler-MacLaurin summation formula

$$\eta(z) = \int_0^1 dw\, \eta(w) + \sum_{m=1}^{\infty} \frac{\eta^{(m-1)}(1) - \eta^{(m-1)}(0)}{m!} B_m(z)\,. \tag{13.24}$$

As we are considering zero–average functions, we have from (13.21), and the fact that Bernoulli polynomials are eigenvectors of the Perron-Frobenius operator

$$C_{\eta,\eta}(n) = \sum_{m=1}^{\infty} \frac{(2^{-m})^n (\eta^{(m)}(1) - \eta^{(m)}(0))}{m!} \int_0^1 dz\, \eta(z) B_m(z)\,.$$

The decomposition (13.24) is also useful to make us realize that the linear functionals ψ_i^* are quite singular objects: if we write it as

$$\eta(z) = \sum_{m=0}^{\infty} B_m(z)\, \psi_m^*[\eta]$$

we see that these functionals are of the form

$$\psi_i^*[\varepsilon] = \int_0^1 dw\, \Psi_i(w) \varepsilon(w)$$

where

$$\Psi_i(w) = \frac{(-1)^{i-1}}{i!} \left(\delta^{(i-1)}(w-1) - \delta^{(i-1)}(w) \right) \tag{13.25}$$

when $i \geq 1$, while $\Psi_0(w) = 1$. Such a representation is only meaningful when the function ε is analytic in $w, w-1$ neighborhoods.

Commentary

Remark 13.1 Surveys of rigorous theory. For a physicist, Driebe's monograph [1.22] might be the most accessible introduction into main theories touched upon in this chapter. We recommend references listed in sect. 1.8 for an introduction into the mathematical literature on this subject. There are a number of reviews of the mathematical approach to dynamical zeta functions and spectral determinants, with pointers to the original references, such as refs. [13.1, 13.2]. An alternative approach to spectral properties of the Perron-Frobenius operator is illustrated in ref. [13.3].

The ergodic theory, as presented by Sinai [13.15] and others, tempts one to describe the densities that the evolution operator acts on in terms of either integrable or square integrable functions. As we have already seen, for our purposes, this space is not suitable. An introduction to ergodic theory is given by Sinai, Kornfeld and Fomin [13.16]; more advanced and more old fashioned presentations are Walters [13.17] and Denker, Grillenberger and Sigmund [13.18]; and a more formal Peterson [13.19].

Remark 13.2 Fredholm theory. Our brief summary of Fredholm theory is based on the exposition in ref. [13.4]. A technical introduction of the theory from an operatorial point of view is contained in ref. [13.5]. The theory has been generalized in ref. [13.6].

Remark 13.3 Bernoulli shift. For a more detailed discussion, consult chapter 17.1 or The extension of Fredholm theory to the case or Bernoulli shift on $C^{k+\alpha}$ (in which the Perron-Frobenius operator is *not* compact technically it is only *quasi-compact*, that is the essential spectral radius is strictly smaller than the spectral radius) has been given by Ruelle [13.7]: a concise and readable statement of the results is contained in ref. [13.8].

Remark 13.4 Higher dimensions and generalized Fredholm theory. When extending Bernoulli shift to higher dimensions. Extensions of Fredholm theory [13.6], which avoid problems with multi-dimensional residue calculus, may be used: see ref. [13.9].

Remark 13.5 Hyperbolic dynamics. When dealing with hyperbolic systems one might try to reduce back to the expanding case by projecting the dynamics along the unstable directions. As mentioned in the text this might be technically quite involved, as usually such the unstable foliation is not characterized by very strong smoothness properties. For such an approach, see ref. [13.3].

Remark 13.6 Spectral determinants for smooth flows. The theorem on p. 169 applies also to hyperbolic analytic maps in d dimensions and smooth hyperbolic analytic flows in $(d+1)$ dimensions, provided that the flow can be reduced to a piecewise analytic map by suspension on a Poincaré section complemented by an analytic "ceiling" function (3.2) which accounts for a variation in the section return times. For example, if we take as the ceiling function $g(x) = e^{sT(x)}$, where $T(x)$ is the time of the next Poincaré section for a trajectory staring at x, we reproduce the flow spectral determinant (12.23). Proofs are getting too hard for the purposes of this chapter; details are discussed in ref.(?).

Remark 13.7 Examples. Examples of analytic hyperbolic maps are provided by small analytic perturbations of the cat map (where the Markov partitioning is non-trivial [13.10]), the 3-disk repeller, and the 2-d baker's map.

Remark 13.8 Explicit diagonalization. For 1-d repellers a diagonalization of an explicit truncated L_{mn} matrix evaluated in a judiciously chosen basis may yield many more eigenvalues than a cycle expansion (see refs. [13.11, 13.12]). The reasons why one persists anyway in using the periodic orbit theory are partially aesthetic, and partially pragmatic. Explicit L_{mn} demands explicit choice of a basis and is thus non-invariant, in contrast to cycle expansions which utilize only the invariant information about the flow. In addition, we usually do not know how to construct L_{mn} for a realistic flow, such as the hyperbolic 3-disk game of pinball flow of sect. 1.3, whereas the periodic orbit formulas are general and straightforward to apply.

Remark 13.9 Perron-Frobenius theorem. A proof of the Perron-Frobenius theorem may be found in ref. [13.13]. For positive transfer operators such theorem has been generalized by Ruelle [13.14].

Remark 13.10 Fried estimates. The form of the fall-off of the coefficients in the $F(z)$ expansion, as $u^{n^{1+1/d}}$, is in agreement with the estimates of Fried [13.20] for the spectral determinants of d-dimensional expanding flows.

Remark 13.11 Axiom A systems. Proofs outlined in sect. 13.3 follow the thesis work of H.H. Rugh [13.9, 13.20, 13.21]. For mathematical introduction to the subject, consult the excellent review by V. Baladi [13.1]. Rigorous treatment is given in refs. [13.9, 13.20,

13.21]. It would take us too far to give and explain the definition of the Axiom A systems (see refs. [13.22, 13.23]). Axiom A implies, however, the existence of a Markov partition of the phase space from which the properties 2 and 3 assumed on p. 165 follow.

Remark 13.12 Exponential mixing speed of the Bernoulli shift. We see from (13.23) that for the Bernoulli shift the exponential decay rate of correlations coincides with the Lyapunov exponent: while such an identity holds for a number of systems, it is by no means a general result, and there exist explicit counterexamples.

Remark 13.13 Left eigenfunctions. We shall never use explicit form of left eigenfunctions, corresponding to highly singular kernels like (13.25). Many details have been elaborated in a number of papers, like ref. [13.24], with a daring physical interpretation.

Remark 13.14 Ulam's idea. The approximation of Perron-Frobenius operator defined by (8.33) has been shown to reproduce correctly the spectrum for expanding maps, once finer and finer Markov partitions are used [13.25]. The subtle point of choosing a phase space partitioning for a "generic case" is discussed in ref. [13.26].

Résumé

A serious theory of cycle expansions requires a deeper understanding of their analyticity and convergence. If we restrict the considerations to those few ideal systems where symbolic dynamics and hyperbolicity can be controlled, it is possible to treat traces and determinants in a rigorous fashion, and beautiful rigorous results about analyticity properties of dynamical zeta functions and spectral determinants outlined above follow.

Most systems of interest are *not* of the "axiom A" category; they are neither purely hyperbolic nor do they have a simple symbolic dynamics grammar. Importance of symbolic dynamics is sometime grossly unappreciated; the crucial ingredient for nice analyticity properties of zeta functions is existence of finite grammar (coupled with uniform hyperbolicity). The dynamical systems that we are *really* interested in - for example, smooth bound Hamiltonian potentials - are presumably never really chaotic, and the central question remains: how to attack the problem in systematic and controllable fashion?

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