Chapter 12

Spectral determinants

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> "It seems very pretty," she said when she had finished it, "but it's rather hard to understand!" (You see she didn't like to confess, even to herself, that she couldn't make it out at all.) "Somehow it seems to fill my head with ideas — only I don't exactly know what they are!"
>
> Lewis Carroll, *Through the Looking Glass*

The problem with trace formulas (11.9), (11.19) and (11.24) is that they diverge at $z = e^{-s_0}$, respectively $s = s_0$, that is, precisely where one would like to use them. While this does not prevent numerical estimation of some "thermodynamic" averages for iterated mappings, in the case of the Gutzwiller trace formula of chapter 27 this leads to a perplexing observation that crude estimates of the radius of convergence seem to put the entire physical spectrum out of reach (see chapter 13). We shall now cure this problem by going from trace formulas to determinants. The idea is illustrated by fig. 1.11: Determinants tend to have larger analyticity domains because if $\mathrm{tr}\, \mathcal{L}/(1-z\mathcal{L}) = -\frac{d}{dz} \ln \det(1-z\mathcal{L})$ diverges at a particular value of z, then $\det(1-z\mathcal{L})$ might have an isolated zero there, and a zero of a function is easier to determine than its radius of convergence.

The eigenvalues of evolution operators are given by the zeros of corresponding determinants, and one way to evaluate determinants is to expand them in terms of traces, using the matrix identity $\log \det = \mathrm{tr} \log$. Traces of evolution operators can be evaluated as integrals over Dirac delta functions, and in this way the spectra of evolution operators become related to periodic orbits.

12.1 Spectral determinants for maps

The eigenvalues z_k of a linear operator are given by the zeros of the determinant

$$\det(1 - z\mathcal{L}) = \prod_k (1 - z/z_k)\,. \tag{12.1}$$

For finite matrices this is the characteristic determinant; for operators this is the Hadamard representation of the *spectral determinant* (here again we spare the reader from pondering possible regularization factors). Consider first the case of maps, for which the evolution operator advances the densities by integer steps in time. In this case we can use the formal matrix identity

$$\ln \det(1 - M) = \mathrm{tr}\, \ln(1 - M) = -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}\, M^n\,, \tag{12.2}$$

to relate the spectral determinant of an evolution operator for a map to its traces (11.7), that is, periodic orbits:

$$\begin{aligned} \det(1 - z\mathcal{L}) &= \exp\left(-\sum_n \frac{z^n}{n} \mathrm{tr}\, \mathcal{L}^n\right) \\ &= \exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{1}{r} \frac{z^{n_p r} e^{r\beta\cdot A_p}}{\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|}\right)\,. \end{aligned} \tag{12.3}$$

Going the other way, the trace formula (11.9) can be recovered from the spectral determinant by taking a derivative

$$\mathrm{tr}\, \frac{z\mathcal{L}}{1 - z\mathcal{L}} = -z \frac{d}{dz} \ln \det(1 - z\mathcal{L})\,. \tag{12.4}$$

fast track:
sect. 12.2, p. 217

Example 12.1 Spectral determinants of transfer operators

For a piecewise-linear map (7.11) with a finite Markov partition, an explicit formula for the spectral determinant follows by substituting the trace formula (11.10) into (12.3):

$$\det(1-z\mathcal{L}) = \prod_{k=0}^{\infty}\left(1-\frac{t_0}{\Lambda_0^k}-\frac{t_1}{\Lambda_1^k}\right) , \tag{12.5}$$

where $t_s = z/|\Lambda_s|$. The eigenvalues are - as they should be - (11.11), the ones that we already determined from the trace formula (11.9).

The exponential spacing of eigenvalues guarantees that the spectral determinant (12.5) is an entire function. It is this property that will generalize to piecewise smooth flows with finite Markov paritions, and single out spectral determinants rather than the trace formulas or dynamical zeta functions as the tool of choice for evaluation of spectra.

12.2 Spectral determinant for flows

> ... an analogue of the [Artin-Mazur] zeta function for diffeomorphisms seems quite remote for flows. However we will mention a wild idea in this direction. $[\cdots]$ define $l(\gamma)$ to be the minimal period of γ $[\cdots]$ then define formally (another zeta function!) $Z(s)$ to be the infinite product
>
> $$Z(s) = \prod_{\gamma\in\Gamma}\prod_{k=0}^{\infty}\left(1-[\exp l(\gamma)]^{-s-k}\right) .$$
>
> Stephen Smale, *Differentiable Dynamical Systems*

We write the formula for the spectral determinant for flows by analogy to (12.3)

$$\det(s-\mathcal{A}) = \exp\left(-\sum_p\sum_{r=1}^{\infty}\frac{1}{r}\frac{e^{r(\beta\cdot A_p - sT_p)}}{\left|\det\left(\mathbf{1}-\mathbf{J}_p^r\right)\right|}\right) , \tag{12.6}$$

and then check that the trace formula (11.19) is the logarithmic derivative of the spectral determinant so defined

$$\operatorname{tr}\frac{1}{s-\mathcal{A}} = \frac{d}{ds}\ln\det(s-\mathcal{A}) . \tag{12.7}$$

To recover $\det(s-\mathcal{A})$ integrate both sides $\int_{s_0}^{s} ds$. With z set to $z=e^{-s}$ as in (11.20), the spectral determinant (12.6) has the same form for both

maps and flows. We shall refer to (12.6) as *spectral determinant*, as the spectrum of the operator $\mathcal{A}$ is given by the zeros of

$$\det(s-\mathcal{A}) = 0 . \tag{12.8}$$

We now note that the r sum in (12.6) is close in form to the expansion of a logarithm. This observation enables us to recast the spectral determinant into an infinite product over periodic orbits as follows:

Let $\mathbf{J}_p$ be the p-cycle $[d\times d]$ transverse Jacobian matrix, with eigenvalues $\Lambda_{p,1}$, $\Lambda_{p,2}$, ..., $\Lambda_{p,d}$. Expanding $1/(1-1/\Lambda_{p,e})$, $1/(1-\Lambda_{p,c})$ in (11.4) as geometric series, substituting back into (12.6), and resumming the logarithms, we find that the spectral determinant is formally given by the infinite product

$$\begin{aligned}
\det(s-\mathcal{A}) &= \prod_{k_1=0}^{\infty}\cdots\prod_{l_c=0}^{\infty}\frac{1}{\zeta_{k_1\cdots l_c}} \\
1/\zeta_{k_1\cdots l_c} &= \prod_p\left(1-t_p\frac{\Lambda_{p,e+1}^{l_1}\Lambda_{p,e+2}^{l_2}\cdots\Lambda_{p,d}^{l_c}}{\Lambda_{p,1}^{k_1}\Lambda_{p,2}^{k_2}\cdots\Lambda_{p,e}^{k_e}}\right) \qquad (12.9)\\
t_p &= t_p(z,s,\beta) = \frac{1}{|\Lambda_p|}e^{\beta\cdot A_p - sT_p}z^{n_p} . \qquad (12.10)
\end{aligned}$$

Here we have inserted a topological cycle length weigth z^{n_p} for reasons which will become apparent in chapter 15; eventually we shall set $z=1$. The observable whose average we wish to compute contributes through the A_p term, which is the p cycle average of the multiplicative weight $e^{A^t(x)}$. By its definition (8.1), for maps the weight is a product along the cycle points

$$e^{A_p} = \prod_{j=0}^{n_p-1} e^{a(f^j(x_p))} ,$$

and for the flows the weight is an exponential of the integral (8.5) along the cycle

$$e^{A_p} = \exp\left(\int_0^{T_p} a(x(\tau))d\tau\right) .$$

This formula is correct for scalar weighting functions; more general matrix valued weights require a time-ordering prescription as in the Jacobian matrix of sect. 4.1.

Now we are finally poised to deal with the problem posed at the beginning of chapter 11; how do we actually evaluate the averages introduced in sect. 8.1? The eigenvalues of the dynamical averaging evolution operator are given by the values of s for which the spectral determinant (12.6) of the evolution operator (8.19) vanishes. If we can compute the leading eigenvalue $s_0(\beta)$ and its derivatives, we are done. Unfortunately, the infinite product formula (12.9) is no more than a shorthand notation for the periodic orbit weights contributing to the spectral determinant; more work will be needed to bring such cycle formulas into a tractable form. This we shall accomplish in chapter 15, but this point in the narrative is a natural point to introduce a still another variant of a determinant, the dynamical zeta function.

12.3 Dynamical zeta functions

It follows from sect. 11.1.1 that if one is interested only in the leading eigenvalue of $\mathcal{L}^t$, the size of the p cycle neighborhood can be approximated by $1/|\Lambda_p|^r$, the dominant term in the $rT_p = t \to \infty$ limit, where $\Lambda_p = \prod_e \Lambda_{p,e}$ is the product of the expanding eigenvalues of the Jacobian matrix $\mathbf{J}_p$. With this replacement the spectral determinant (12.6) is replaced by the *dynamical zeta function*

$$1/\zeta \;=\; \exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{1}{r} t_p^r\right) \tag{12.11}$$

that we have already derived heuristically in sect. 1.5.2. Resumming the logarithms using $\sum_r t_p^r/r = -\ln(1-t_p)$ we obtain the *Euler product rep.* of the dynamical zeta function:

$$1/\zeta \;=\; \prod_p (1-t_p)\ . \tag{12.12}$$

For reasons of economy of the notation, we shall usually omit the explicit dependence of $1/\zeta$, t_p on z, s, β whenever the dependence is clear from the context.

The approximate trace formula (11.24) plays the same role vis-a-vis the dynamical zeta function

$$\Gamma(s) = \frac{d}{ds} \ln \zeta^{-1} = \sum_p \frac{T_p t_p}{1-t_p}\,, \tag{12.13}$$

as the exact trace formula (11.19) plays vis-a-vis the spectral determinant (12.6), see (12.7). The heuristically derived dynamical zeta function of sect. 1.5.2 now re-emerges as the $1/\zeta_{0\ldots0}(z)$ part of the *exact* spectral determinant; other factors in the infinite product (12.9) affect the non-leading eigenvalues of $\mathcal{L}$.

To summarize: the dynamical zeta function (12.12) associated with the flow $f^t(x)$ is defined as the product over all prime cycles p. T_p, n_p and Λ_p are the period, topological length and stability of prime cycle p, A_p is the integrated observable $a(x)$ evaluated on a single traversal of cycle p (see (8.5)), s is a variable dual to the time t, z is dual to the discrete "topological" time n, and $t_p(z,s,\beta)$ is the local trace over the cycle p. We have included the factor z^{n_p} in the definition of the cycle weight in order to keep track of the number of times a cycle traverses the surface of section. The dynamical zeta function is useful because

$$1/\zeta(s) = 0 \tag{12.14}$$

vanishes at s equal to s_0, the leading eigenvalue of $\mathcal{L}^t = e^{t\mathcal{A}}$, and often the leading eigenvalue is all that is needed in applications. The above completes our derivation of the trace and determinant formulas for classical chaotic flows. In chapters that follow we shall make these formulas tangible by working out a series of simple examples.

The remainder of this chapter offers examples of zeta functions.

fast track:
chapter 15, p. 279

12.3.1 A contour integral formulation

The following observation is sometimes useful, in particular when the zeta functions have richer analytic structure than just zeros and poles, as in the case of intermittency (chapter 18): Γ_n, the trace sum (11.22), can be expressed in terms of the dynamical zeta function (12.12)

$$1/\zeta(z) = \prod_p \left(1 - \frac{z^{n_p}}{|\Lambda_p|}\right) \quad . \tag{12.15}$$

as a contour integral

$$\Gamma_n = \frac{1}{2\pi i} \oint_{\gamma_r^-} z^{-n} \left(\frac{d}{dz} \log \zeta^{-1}(z)\right) dz \quad , \tag{12.16}$$

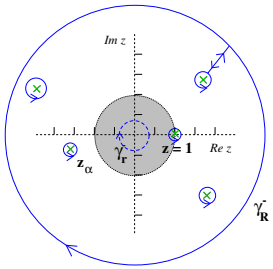

Figure 12.1: The survival probability Γ_n can be split into contributions from poles (x) and zeros (o) between the small and the large circle and a contribution from the large circle.

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where a small contour γ_r^- encircles the origin in negative (clockwise) direction. If the contour is small enough, that is it lies inside the unit circle $|z| = 1$, we may write the logarithmic derivative of $\zeta^{-1}(z)$ as a convergent sum over all periodic orbits. Integrals and sums can be interchanged, the integrals can be solved term by term, and the trace formula (11.22) is recovered. For hyperbolic maps, cycle expansion or other techniques provide an analytic extension of the dynamical zeta function beyond the leading zero; we may therefore deform the orignal contour into a larger circle with radius R which encircles both poles and zeros of $\zeta^{-1}(z)$, see fig. 12.1. Residue calculus turns this into a sum over the zeros z_α and poles z_β of the dynamical zeta function, that is

$$\Gamma_n = \sum_{|z_\alpha|<R}^{\text{zeros}} \frac{1}{z_\alpha^n} - \sum_{|z_\beta|<R}^{\text{poles}} \frac{1}{z_\beta^n} + \frac{1}{2\pi i} \oint_{\gamma_R^-} dz\, z^{-n} \frac{d}{dz} \log \zeta^{-1}, \tag{12.17}$$

where the last term gives a contribution from a large circle γ_R^-. We thus find exponential decay of Γ_n dominated by the leading zero or pole of $\zeta^{-1}(z)$.

12.3.2 Dynamical zeta functions for transfer operators

Ruelle's original dynamical zeta function was a generalization of the topological zeta function (10.21) that we shall discuss in chapter 10 to a function that assigns different weights to different cycles:

$$\zeta(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \left(\sum_{x_i \in \mathrm{Fix} f^n} \mathrm{tr} \prod_{j=0}^{n-1} g(f^j(x_i)) \right) .$$

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Here the sum goes over all periodic points x_i of period n, and $g(x)$ is any (matrix valued) weighting function, with weight evaluated multiplicatively along the trajectory of x_i.

By the chain rule (4.30) the stability of any n-cycle of a 1-d map is given by $\Lambda_p = \prod_{j=1}^n f'(x_i)$, so the 1-$d$ map cycle stability is the simplest example of a multiplicative cycle weight $g(x_i) = f'(x_i)$, and indeed - via the Perron-Frobenius evolution operator (7.9) - the historical motivation for Ruelle's more abstract construction.

In particular, for a piecewise-linear map with a finite Markov partition, the dynamical zeta function is given by a finite polynomials, a straightforward generalization of determinant of the topological transition matrix (9.2). As explained in sect. 10.3, for a finite $[N\times N]$ dimensional matrix the determinant is given by

$$\prod_p (1 - t_p) = \sum_{n=1}^{N} z^n c_n \,,$$

where c_n is given by the sum over all non-self-intersecting closed paths of length n together with products of all non-intersecting closed paths of total length n. We illustrate this by the piecewise linear repeller (7.11).

Example 12.2 Piecewise linear repeller

Due to the piecewise linearity, the stability of any n-cycle factorizes as $\Lambda_{s_1 s_2 \ldots s_n} = \Lambda_0^m \Lambda_1^{n-m}$, where m is total number of times letter $s_j = 0$ appears in the p symbol sequence, so the traces in the sum (11.24) are of a particularly simple form

$$\mathrm{tr}\, T^n = \Gamma_n = \left(\frac{1}{|\Lambda_0|} + \frac{1}{|\Lambda_1|} \right)^n .$$

The dynamical zeta function (12.11) evaluated by resumming the traces

$$1/\zeta(z) = 1 - z/|\Lambda_0| - z/|\Lambda_1| \tag{12.18}$$

is indeed the determinant $\det(1 - zT)$ *of the transfer operator (7.13), almost as simple as the topological zeta function (10.25).*

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More generally, piecewise-linear approximations to dynamical systems yield polynomial or rational polynomial cycle expansions, provided that the symbolic dynamics is a subshift of finite type (see sect. 9.7).

We see that the exponential proliferation of cycles so dreaded by quantum chaoticists is a bogus anxiety; we are dealing with exponentially many cycles of increasing length and instability, but all that really matters in this example are the stabilities of the two fixed points. Clearly the information carried by the infinity of longer cycles is highly redundant; we shall learn in chapter 15 how to exploit systematically this redundancy.

12.4 False zeros

Compare (12.18) with the Euler product (12.12). For simplicity take the two scales equal, $|\Lambda_0| = |\Lambda_1| = e^{\lambda}$. Our task is to determine the leading zero $z = e^{\gamma}$ of the Euler product. It is a novice error to assume that the infinite Euler product (12.12) vanishes whenever one of its factors vanishes. If that were true, each factor $(1 - z^{n_p}/|\Lambda_p|)$ would yield

$$0 = 1 - e^{n_p(\gamma - \lambda_p)}, \tag{12.19}$$

that is the escape rate γ would equal the stability exponent of a repulsive fixed point. False! The exponentially growing number of cycles with growing period conspires to shift the zeros of the infinite product. The correct formula follows from (12.18)

$$0 = 1 - e^{\gamma - \lambda + h}, \qquad h = \ln 2. \tag{12.20}$$

This particular formula for the escape rate is a special case of a general relation between escape rates, Lyapunov exponents and entropies that is not yet included into this book. The physical interpretation is that the escape induced by repulsion by each unstable fixed point is diminished by the rate of backscatter from other repelling segments, that is the entropy h; the positive entropy of orbits of the same stability shifts the "false zeros" $z = e^{\lambda_p}$ of the Euler product (12.12) to the true zero $z = e^{\lambda - h}$.

12.5 More examples of spectral determinants

For expanding 1-d mappings the spectral determinant (12.9) takes form

$$\det(s - \mathcal{A}) = \prod_p \prod_{k=0}^{\infty} \left(1 - t_p/\Lambda_p^k\right), \qquad t_p = \frac{e^{\beta A_p - sT_p}}{|\Lambda_p|} z^{n_p}. \tag{12.21}$$

For a periodic orbit of a 2-dimensional hyperbolic Hamiltonian flow with one expanding transverse eigenvalue Λ, $|\Lambda| > 1$, and one contracting transverse eigenvalue $1/\Lambda$, the weight in (11.4) is expanded as follows:

$$\frac{1}{\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|} = \frac{1}{|\Lambda|^r (1 - 1/\Lambda_p^r)^2} = \frac{1}{|\Lambda|^r} \sum_{k=0}^{\infty} \frac{k+1}{\Lambda_p^{kr}}. \tag{12.22}$$

The spectral determinant exponent can be resummed,

$$-\sum_{r=1}^{\infty} \frac{1}{r} \frac{e^{(\beta A_p - sT_p)r}}{\left|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)\right|} = \sum_{k=0}^{\infty} (k+1) \log\left(1 - \frac{e^{\beta A_p - sT_p}}{|\Lambda_p| \Lambda_p^k}\right),$$

and the spectral determinant for a 2-dimensional hyperbolic Hamiltonian flow rewritten as an infinite product over prime cycles

$$\det(s - \mathcal{A}) = \prod_p \prod_{k=0}^{\infty} \left(1 - t_p/\Lambda_p^k\right)^{k+1}. \tag{12.23}$$

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In such formulas, t_p is a weight associated with the p cycle (letter t refers to the "local trace" evaluated along the p cycle trajectory), and the index p runs through all distinct prime cycles. We use z as a formal parameter which keeps track of the topological cycle lengths, to assist us in expanding zeta functions and determinants, then set it to $z = 1$ in calculations.

12.5.1 Spectral determinants *vs.* dynamical zeta functions

In sect. 12.3 we derived the dynamical zeta function as an approximation to the spectral determinant. Here we relate dynamical zeta functions to the spectral determinants *exactly*, by showing that a dynamical zeta function can be expressed as a ratio of products of spectral determinants.

The elementary identity for d-dimensional matrices

$$1 = \frac{1}{\det(1 - \mathbf{J})} \sum_{k=0}^{d} (-1)^k \mathrm{tr}\left(\wedge^k \mathbf{J}\right), \tag{12.24}$$

inserted into the exponential representation (12.11) of the dynamical zeta function, relates the dynamical zeta function to *weighted* spectral determinants. For 1-d maps the identity

$$1 = \frac{1}{(1 - 1/\Lambda)} - \frac{1}{\Lambda} \frac{1}{(1 - 1/\Lambda)}$$

substituted into (12.11) yields an expression for the dynamical zeta function for 1-d maps as a ratio of two spectral determinants

$$1/\zeta = \frac{\det(1 - \mathcal{L})}{\det(1 - \mathcal{L}_{(1)})} \tag{12.25}$$

where the cycle weight in $\mathcal{L}_{(1)}$ is given by replacement $t_p \to t_p/\Lambda_p$. As we shall see in chapter 13, this establishes that for nice hyperbolic flows $1/\zeta$ is meromorphic, with poles given by the zeros of $\det(1-\mathcal{L}_{(1)})$. The dynamical zeta function and the spectral determinant have the same zeros - only in exceptional circumstances some zeros of $\det(1-\mathcal{L}_{(1)})$ might be cancelled by coincident zeros of $\det(1-\mathcal{L}_{(1)})$. Hence even though we have derived the dynamical zeta function in sect. 12.3 as an "approximation" to the spectral determinant, the two contain the same spectral information.

For 2-dimensional Hamiltonian flows the above identity yields

$$\frac{1}{|\Lambda|} = \frac{1}{|\Lambda|(1-1/\Lambda)^2}(1-2/\Lambda+1/\Lambda^2)\,,$$

so

$$1/\zeta = \frac{\det(1-\mathcal{L})\det(1-\mathcal{L}_{(2)})}{\det(1-\mathcal{L}_{(1)})}\,. \tag{12.26}$$

This establishes that for nice hyperbolic flows dynamical zeta function is meromorphic in 2-d.

Example 12.3 Dynamical zeta functions for 2-d Hamiltonian flows

The relation (12.26) is not particularly useful for our purposes. Instead we insert the identity

$$1 = \frac{1}{(1-1/\Lambda)^2} - \frac{2}{\Lambda}\frac{1}{(1-1/\Lambda)^2} + \frac{1}{\Lambda^2}\frac{1}{(1-1/\Lambda)^2}$$

into the exponential representation (12.11) of $1/\zeta_k$, and obtain

$$1/\zeta_k = \frac{F_k F_{k+2}}{F_{k+1}^2}\,. \tag{12.27}$$

Even though we have no guarantee that F_k are entire, we do know that the upper bound on the leading zeros of F_{k+1} lies strictly below the leading zeros of F_k, and therefore we expect that for 2-dimensional Hamiltonian flows the dynamical zeta function $1/\zeta_k$ has generically a double *leading pole coinciding with the leading zero of the F_{k+1} spectral determinant. This might fail if the poles and leading eigenvalues come in wrong order, but we have not encountered such situation in our numerical investigations. This result can also be stated as follows: the theorem that establishes that the spectral determinant (12.23) is entire, implies that the poles in $1/\zeta_k$ must have right multiplicities in order that they be cancelled in the $F = \prod 1/\zeta_k^{k+1}$ product.*

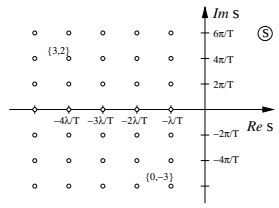

Figure 12.2: The classical resonances $\alpha = \{k,n\}$ for a 2-disk game of pinball, equation (12.28).

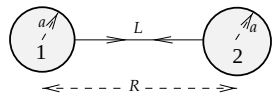

Figure 12.3: A game of pinball consisting of two disks of equal size in a plane, with its only periodic orbit. (A. Wirzba)

12.6 All too many eigenvalues?

What does the 2-dimensional hyperbolic Hamiltonian flow spectral determinant (12.23) tell us? Consider one of the simplest conceivable hyperbolic flows: the game of pinball of fig. 12.3 consisting of two disks of equal size in a plane. There is only one periodic orbit, with the period T and the expanding eigenvalue Λ is given by elementary considerations (see exercise 5.5), and the resonances $\det(s_\alpha - \mathcal{A}) = 0$, $\alpha = \{k,n\}$ plotted in fig. 12.2

$$s_\alpha = -(k+1)\lambda + n\frac{2\pi i}{T}\,, \quad n\in\mathbb{Z}\,,\, k\in\mathbb{Z}_+\,, \quad \text{multiplicity } k+1\,, \tag{12.28}$$

can be read off the spectral determinant (12.23) for a single unstable cycle:

$$\det(s-\mathcal{A}) = \prod_{k=0}^{\infty}\left(1-e^{-sT}/|\Lambda|\Lambda^k\right)^{k+1}\,. \tag{12.29}$$

In the above $\lambda = \ln|\Lambda|/T$ is the cycle Lyapunov exponent. For an open system, the real part of the eigenvalue s_α gives the decay rate of αth eigenstate, and the imaginary part gives the "node number" of the eigenstate. The negative real part of s_α indicates that the resonance is unstable, and the decay rate in this simple case (zero entropy) equals to the cycle Lyapunov exponent.

Fast decaying eigenstates with large negative $\mathrm{Re}\, s_\alpha$ are not a problem, but as there are eigenvalues arbitrarily far in the imaginary direction, this might seem like all too many eigenvalues. However, they are necessary -

we can check this by explicit computation of the right hand side of (11.19), the trace formula for flows:

$$\begin{aligned}\sum_{\alpha=0}^{\infty} e^{s_\alpha t} &= \sum_{k=0}^{\infty}\sum_{n=-\infty}^{\infty}(k+1)e^{(k+1)\lambda t+i2\pi nt/T} \\ &= \sum_{k=0}^{\infty}(k+1)\left(\frac{1}{|\Lambda|\Lambda^k}\right)^{t/T}\sum_{n=-\infty}^{\infty}e^{i2\pi n/T} \\ &= \sum_{k=0}^{\infty}\frac{k+1}{|\Lambda|^r\Lambda^{kr}}\sum_{r=-\infty}^{\infty}\delta(r-t/T) \\ &= T\sum_{r=-\infty}^{\infty}\frac{\delta(t-rT)}{|\Lambda|(1-1/\Lambda^r)^2} \qquad (12.30)\end{aligned}$$

So the two sides of the trace formula (11.19) check. The formula is fine for $t > 0$; for $t \to 0_+$ both sides are divergent and need regularization.

The reason why such sums do not occur for maps is that for discrete time we work in the variable $z = e^s$, an infinite strip along Im s maps into an anulus in the complex z plane, and the Dirac delta sum in the above is replaced by the Kronecker delta sum in (11.7). In case at hand there is only one time scale T, and we could as well replace s by variable $z = e^{-s/T}$. In general the flow has a continuum of cycle periods, and the resonance arrays are more irregular, *cf.* fig. 15.1.

Commentary

Remark 12.1 Piecewise monotone maps. A partial list of cases for which the transfer operator is well defined: expanding Hölder case, weighted subshifts of finite type, expanding differentiable case, see Bowen [1.13]: expanding holomorphic case, see Ruelle [13.9]; piecewise monotone maps of the interval, see Hofbauer and Keller [12.15] and Baladi and Keller [12.18].

Remark 12.2 Smale's wild idea. Smale's wild idea quoted on page 217 was technically wrong because 1) the Selberg zeta yields the spectrum of a quantum mechanical Laplacian rather than the classical resonances, 2) the spectral determinant weights are different from what Smale conjectured, as the individual cycle weights also depend on the stability of the cycle, 3) the formula is not dimensionally correct, as k is an integer and s is dimensionally inverse time. Only

for spaces of constant negative curvature do all cycles have the same Lyapunov exponent $\lambda = \ln|\Lambda_p|/T_p$. In this case normalizing the time so that $\lambda = 1$ the factors e^{-sT_p}/Λ_p^k in (12.9) simplify to $s^{-(s+k)T_p}$, as intuited in Smale's wild idea quoted on page 217 (where $l(\gamma)$ is the cycle period denoted here by T_p). Nevertheless, Smale's intuition was remarkably on the target.

Remark 12.3 Is this a generalization of the Fourier analysis? The Fourier analysis is a theory of the space $\leftrightarrow$ eignfunctions duality for dynamics on a circle. The sense in which the periodic orbit theory is the generalization of the Fourier analysis to nonlinear flows is discussed in ref. [12.4], a very readable introduction to the Selberg Zeta function.

Remark 12.4 Zeta functions, antecedents. For a function to be deserving of the appellation "zeta function", one expects it to have an Euler product (12.12) representation, and perhaps also satisfy a functional equation. Various kinds of zeta functions are reviewed in refs. [12.8, 12.9, 12.10]. Historical antecedents of the dynamical zeta function are the fixed-point counting functions introduced by Weil [12.11], Lefschetz [12.12] and Artin and Mazur [12.13], and the determinants of transfer operators of statistical mechanics [1.14].

In his review article Smale [1.12] already intuited, by analogy to the Selberg Zeta function, that the spectral determinant is the right generalization for continuous time flows. In dynamical systems theory dynamical zeta functions arise naturally only for piecewise linear mappings; for smooth flows the natural object for study of classical and quantal spectra are the spectral determinants. Ruelle had derived the relation (12.3) between spectral determinants and dynamical zeta functions, but as he was motivated by the Artin-Mazur zeta function (10.21) and the statistical mechanics analogy, he did not consider the spectral determinant a more natural object than the dynamical zeta function. This has been put right in papers on "flat traces" [9.22, 13.27].

The nomenclature has not settled down yet; what we call evolution operators here is called transfer operators [1.16], Perron-Frobenius operators [12.6] and/or Ruelle-Araki operators elsewhere. Here we refer to kernels such as (8.19) as evolution operators. We follow Ruelle in usage of the term "dynamical zeta function", but elsewhere in the literature function (12.12) is often called the Ruelle zeta function. Ruelle [1.17] points out the corresponding transfer operator T was never considered by either Perron or Frobenius; a more appropriate designation would be the Ruelle-Araki operator. Determinants similar to or identical with our spectral determinants are sometimes called Selberg Zetas, Selberg-Smale zetas [1.4], functional determinants, Fredholm determinants, or even - to maximize confusion - dynamical zeta functions [12.14]. A Fredholm determinant is a notion that applies only

to the trace class operators - as we consider here a somewhat wider class of operators, we prefer to refer to their determinants losely as "spectral determinants".

Résumé

The spectral problem is now recast into a problem of determining zeros of either the *spectral determinant*

$$\det(s-\mathcal{A}) = \exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{1}{r} \frac{e^{(\beta\cdot A_p - sT_p)r}}{\left|\det\left(\mathbf{1}-\mathbf{J}_p^r\right)\right|}\right),$$

or the leading zeros of the *dynamical zeta function*

$$1/\zeta = \prod_p (1-t_p), \qquad t_p = \frac{1}{|\Lambda_p|} e^{\beta\cdot A_p - sT_p}.$$

The spectral determinant is the tool of choice in actual calculations, as it has superior convergence properties (this will be discussed in chapter 13 and is illustrated, for example, by table 15.2). In practice both spectral determinants and dynamical zeta functions are preferable to trace formulas because they yield the eigenvalues more readily; the main difference is that while a trace diverges at an eigenvalue and requires extrapolation methods, determinants vanish at s corresponding to an eigenvalue s_α, and are analytic in s in an open neighborhood of s_α.

The critical step in the derivation of the periodic orbit formulas for spectral determinants and dynamical zeta functions is the hyperbolicity assumption, that is the assumption that all cycle stability eigenvalues are bounded away from unity, $|\Lambda_{p,i}| \neq 1$. By dropping the prefactors in (1.4), we have given up on any possibility of recovering the precise distribution of starting x (return to the past is rendered moot by the chaotic mixing and the exponential growth of errors), but in exchange we gain an effective description of the asymptotic behavior of the system. The pleasant surprise (to be demonstrated in chapter 15) is that the infinite time behavior of an unstable system turns out to be as easy to determine as its short time behavior.

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