Exercises

Exercise 8.1 **How unstable is the Hénon attractor?**

(a) *Evaluate numerically the Lyapunov exponent by iterating the Hénon map*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 - ax^2 + y \\ bx \end{bmatrix}$$

for $a = 1.4$, $b = 0.3$.

(b) *Now check how robust is the Lyapunov exponent for the Hénon attractor? Evaluate numerically the Lyapunov exponent by iterating the Hénon map for* $a = 1.39945219$, $b = 0.3$. *How much do you trust now your result for the part (a) of this exercise?*

Exercise 8.2 **Expectation value of a vector observable and its moments.** *Check and extend the expectation value formulas (8.12) by evaluating the derivatives of* $s(\beta)$ *up to 4-th order for the space average* $\langle \exp(\beta \cdot A^t) \rangle$ *with* a_i *a vector quantity:*

(a)

$$\left.\frac{\partial s}{\partial \beta_i}\right|_{\beta=0} = \lim_{t\to\infty} \frac{1}{t} \left\langle A_i^t \right\rangle = \langle a_i \rangle \,, \tag{8.35}$$

(b)

$$\begin{aligned} \left.\frac{\partial^2 s}{\partial \beta_i \partial \beta_j}\right|_{\beta=0} &= \lim_{t\to\infty} \frac{1}{t} \left(\left\langle A_i^t A_j^t \right\rangle - \left\langle A_i^t \right\rangle \left\langle A_j^t \right\rangle \right) \\ &= \lim_{t\to\infty} \frac{1}{t} \left\langle (A_i^t - t \langle a_i \rangle)(A_j^t - t \langle a_j \rangle) \right\rangle \,. \end{aligned} \tag{8.36}$$

Note that the formalism is cmart: it automatically yields the variance *from the mean, rather than simply the 2nd moment* $\langle a^2 \rangle$.

(c) *compute the third derivative of* $s(\beta)$.

(d) *compute the fourth derivative assuming that the mean in (8.35) vanishes,* $\langle a_i \rangle = 0$. *The 4-th order moment formula*

$$K(t) = \frac{\left\langle x^4(t) \right\rangle}{\langle x^2(t) \rangle^2} - 3 \tag{8.37}$$

that you have derived is known as kurtosis*: it measures a deviation from what the 4-th order moment would be were the distribution a pure gaussian (see (20.22) for a concrete example). If the observable is a vector, the kurtosis is given by*

$$K(t) = \frac{\sum_{ij}\left[\langle A_i A_i A_j A_j\rangle + 2\left(\langle A_i A_j\rangle\langle A_j A_i\rangle - \langle A_i A_i\rangle\langle A_j A_j\rangle\right)\right]}{\left(\sum_i \langle A_i A_i\rangle\right)^2} \tag{8.38}$$