Exercises

Exercise 5.1 **A pinball simulator.** *Implement the disk $\rightarrow$ disk maps to compute a trajectory of a pinball for a given starting point, and a given $R{:}a$ = (center-to-center distance):(disk radius) ratio for a 3-disk system. As this requires only computation of intersections of lines and circles together with specular reflections, implementation should be within reach of a high-school student. Please start working on this program now; it will be continually expanded in chapters to come, incorporating the Jacobian calculations, Newton root–finding, and so on.*

Fast code will use elementary geometry (only one $\sqrt{\cdots}$ per iteration, rest are multiplications) and eschew trigonometric functions. Provide a graphic display of the trajectories and of the Poincaré section iterates. To be able to compare with the numerical results of coming chapters, work with $R{:}a = 6$ and/or 2.5 values. Draw the correct versions of fig. 1.8 or fig. 9.4 for $R{:}a =$ 2.5 and/or 6.

Exercise 5.2 **Trapped orbits.** *Shoot 100,000 trajectories from one of the disks, and trace out the strips of fig. 1.8 for various $R{:}a$ by color coding the initial points in the Poincaré section by the number of bounces preceeding their escape. Try also $R{:}a = 6{:}1$, though that might be too thin and require some magnification. The initial conditions can be randomly chosen, but need not - actually a clearer picture is obtained by systematic scan through regions of interest.*

Exercise 5.3 **Pinball stability.** *Add to your exercise 5.1 pinball simulator a routine that computes the the [2$\times$x2] Jacobian matrix. To be able to compare with the numerical results of coming chapters, work with $R{:}a = 6$ and/or 2.5 values.*

Exercise 5.4 **Stadium billiard.** *Consider the* Bunimovich stadium *[3.16, 3.17] defined in fig. 5.2. The Jacobian matrix associated with the reflection is given by (5.21). Here we take $\rho_k = -1$ for the semicircle sections of the boundary, and $\cos\phi_k$ remains constant for all bounces in a rotation sequence. The time of flight between two semicircle bounces is $\tau_k = 2\cos\phi_k$. The Jacobian matrix of one semicircle reflection folowed by the flight to the next bounce is*

$$\mathbf{J} = (-1)\begin{pmatrix} 1 & 2\cos\phi_k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2/\cos\phi_k & 1 \end{pmatrix} = (-1)\begin{pmatrix} -3 & 2\cos\phi_k \\ 2/\cos\phi_k & 1 \end{pmatrix}.$$

A shift must always be followed by $k = 1, 2, 3, \cdots$ bounces along a semicircle, hence the natural symbolic dynamics for this problem is n-ary, with the corresponding Jacobian matrix given by shear (ie. the eigenvalues remain equal to 1 throughout the whole rotation), and k bounces inside a circle lead to

$$\mathbf{J}^k = (-1)^k\begin{pmatrix} -2k-1 & 2k\cos\phi \\ 2k/\cos\phi & 2k-1 \end{pmatrix}. \tag{5.23}$$

The Jacobian matrix of a cycle p of length n_p is given by

$$\mathbf{J}_p = (-1)^{\sum n_k} \prod_{k=1}^{n_p} \begin{pmatrix} 1 & \tau_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ n_k r_k & 1 \end{pmatrix} . \tag{5.24}$$

Adopt your pinball simulator to the Bunimovich stadium.

Exercise 5.5 **Fundamental domain fixed points.** *Use the formula (5.22) for billiard Jacobian matrix to compute the periods T_p and the expanding eigenvalues Λ_p of the fundamental domain $\overline{0}$ (the 2-cycle of the complete 3-disk space) and $\overline{1}$ (the 3-cycle of the complete 3-disk space) fixed points:*

	T_p	Λ_p
$\overline{0}$:	$R-2$	$R-1+R\sqrt{1-2/R}$
$\overline{1}$:	$R-\sqrt{3}$	$-\frac{2R}{\sqrt{3}}+1-\frac{2R}{\sqrt{3}}\sqrt{1-\sqrt{3}/R}$

(5.25)

We have set the disk radius to $a=1$.

Exercise 5.6 **Fundamental domain 2-cycle.** *Verify that for the $\overline{10}$-cycle the cycle length and the trace of the Jacobian matrix are given by*

$$\begin{aligned} L_{10} &= 2\sqrt{R^2-\sqrt{3}R+1}-2, \\ \operatorname{tr}\mathbf{J}_{10} &= 2L_{10}+2+\frac{1}{2}\frac{L_{10}(L_{10}+2)^2}{\sqrt{3}R/2-1} . \end{aligned} \tag{5.26}$$

The $\overline{10}$-cycle is drawn in fig. 9.5. The unstable eigenvalue Λ_{10} follows from (4.16).

Exercise 5.7 **A test of your pinball simulator.** *Test your exercise 5.3 pinball simulator by comparing what it yields with the analytic formulas of exercise 5.5 and 5.6.*

Exercise 5.8 **Birkhoff coordinates.** *Prove that the Birkhoff coordinates are phase-space volume preserving. Hint: compute the determinant of (5.22).*