Exercises

Exercise 14.1 Cycles of the Ulam map. *Test your cycle-searching routines by computing a bunch of short cycles and their stabilities for the Ulam map*

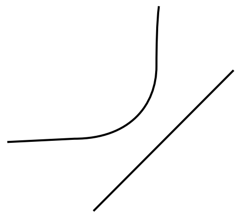
$$f(x) = 4x(1-x)\,. \tag{14.21}$$

Exercise 14.2 Cycles stabilities for the Ulam map, exact. *In exercise 14.1 you should have observed that the numerical results for the cycle stability eigenvalues (4.30) are exceptionally simple: the stability eigenvalue of the $x_0 = 0$ fixed point is 4, while the eigenvalue of any other n-cycle is $\pm 2^n$. Prove this. (Hint: the Ulam map can be conjugated to the tent map (9.10). This problem is perhaps too hard, but give it a try - the answer is in many introductory books on nolinear dynamics.)*

Exercise 14.3 Stability of billiard cycles. *Compute stabilities of few simple cycles.*

(a) *A simple scattering billiard is the two-disk billiard. It consists of a disk of radius one centered at the origin and another disk of unit radius located at $L+2$. Find all periodic orbits for this system and compute their stabilities. (You might have done this already in exercise 1.2; at least now you will be able to see where you went wrong when you knew nothing about cycles and their extraction.)*

(b) *Find all periodic orbits and stabilities for a billiard ball bouncing between the diagonal $y = x$ and one of the hyperbola branches $y = 1/x$.*



Exercise 14.4 Cycle stability. *Add to the pinball simulator of exercise 5.1 a routine that evaluates the expanding eigenvalue for a given cycle.*

Exercise 14.5 Pinball cycles. *Determine the stability and length of all fundamental domain prime cycles of the binary symbol string lengths up to 5 (*or longer*) for $R : a = 6$ 3-disk pinball.*

Exercise 14.6 Newton-Raphson method. *Implement the Newton-Raphson method in 2-d and apply it to determination of pinball cycles.*

Exercise 14.7 Rössler system cycles. *(continuation of exercise 2.7, and exercise 3.1) Determine all cycles up to 5 Poincaré sections returns for the Rössler system (2.11), as well as their stabilities. (Hint: implement (14.16), the multipoint shooting methods for flows; you can cross-check your shortest cycles against the ones listed in table 14.1.)*

n_p	p	y_p	z_p	Λ_e
1	1	6.091768319056803	1.299731937639821	-2.4039535318268
2	01	3.915804049621049	3.692833386542665	-3.5120069815161
3	001	2.278281031720258	7.416480984019008	-2.3419235232340
	011	2.932877559129124	5.670805943881501	5.3449081538885
4	0111	3.466758713211455	4.506217531477667	-16.6967406980700
5	01011	4.162798782914948	3.303903338609633	-23.1995830097831
	01111	3.278914359770783	4.890452922955567	36.8863297988981
6	001011	2.122093931936202	7.886172854283211	-6.8576654190825
	010111	4.059210605826523	3.462265228606606	61.6490940089068
	011111	3.361494458061049	4.718206217035575	-92.0825560711089
7	0101011	3.842769382372052	3.815493592299824	77.7611048852412
	0110111	3.025956697151134	5.451444475664179	-95.1838846735358
	0101111	4.102255295518855	3.395643547170646	-142.2379888163439
	0111111	3.327986189581191	4.787462810306583	218.0283602810993

Table 14.1: The Rössler system (2.11): The itinerary p, a periodic point $x_p = (0, y_p, z_p)$ and the expanding eigenvalue Λ_p for all cycles up to the topological length 7. (Joachim Mathiesen)

Exercise 14.8 **Cycle stability, helium.** *Add to the helium integrator of exercise 2.9 a routine that evaluates the expanding eigenvalue for a given cycle.*

Exercise 14.9 **Colinear helium cycles.** *Determine the stability and length of all fundamental domain prime cycles up to symbol sequence length 5 or longer for collinear helium of fig. 29.5.*

Exercise 14.10 **Uniqueness of unstable cycles***.** *Prove that there exists only one 3-disk prime cycle for a given finite admissible prime cycle symbol string. Hints: look at the Poincaré section mappings; can you show that there is exponential contraction to a unique periodic point with a given itinerary? Exercise 25.1 might be helpful in this effort.*

Exercise 14.11 **Inverse iteration method for a Hamiltonian repeller.** *For the complete repeller case (all binary sequences are realized), the cycles are evaluated as follows. According to (3.10), the coordinates of a periodic orbit of length n_p satisfy the equation*

$$x_{p,i+1} + x_{p,i-1} = 1 - a x_{p,i}^2 \,, \quad i = 1, ..., n_p \,, \tag{14.22}$$

with the periodic boundary condition $x_{p,0} = x_{p,n_p}$. In the complete repeller case, the Hénon map is a realization of the Smale horseshoe, and the symbolic dynamics has a very simple description in terms of the binary alphabet $\epsilon \in \{0,1\}$, $\epsilon_{p,i} = (1 + S_{p,i})/2$, where $S_{p,i}$ are the signs of the corresponding cycle point coordinates, $S_{p,i} = \sigma_{x_{p,i}}$. We start with a preassigned sign sequence $S_{p,1}, S_{p,2}, \ldots, S_{p,n_p}$, and a good initial guess for the coordinates $x'_{p,i}$. Using the inverse of the equation (14.22)

$$x''_{p,i} = S_{p,i} \sqrt{\frac{1 - x'_{p,i+1} - x'_{p,i-1}}{a}} \,, \quad i = 1, ..., n_p \tag{14.23}$$

p	Λ_p	$\sum x_{p,i}$
0	0.71516752438×10^1	-0.6076252185107
1	-0.29528463259×10^1	0.2742918851774
10	-0.98989794855×10^1	0.3333333333333
100	-0.13190727397×10^3	-0.2060113295833
110	0.55896964996×10^2	0.5393446629166
1000	-0.10443010730×10^4	-0.8164965809277
1100	0.57799826989×10^4	0.0000000000000
1110	-0.10368832509×10^3	0.8164965809277
10000	-0.76065343718×10^4	-1.4260322065792
11000	0.44455240007×10^4	-0.6066540777738
10100	0.77020248597×10^3	0.1513755016405
11100	-0.71068835616×10^3	0.2484632276044
11010	-0.58949885284×10^3	0.8706954728949
11110	0.39099424812×10^3	1.0954854155465
100000	-0.54574527060×10^5	-2.0341342556665
110000	0.32222060985×10^5	-1.2152504370215
101000	0.51376165109×10^4	-0.4506624359329
111000	-0.47846146631×10^4	-0.3660254037844
110100	-0.63939998436×10^4	0.3333333333333
101100	-0.63939998436×10^4	0.3333333333333
111100	0.39019387269×10^4	0.5485837703548
111010	0.10949094597×10^4	1.1514633582661
111110	-0.10433841694×10^4	1.3660254037844

Table 14.2: All periodic orbits up to 6 bounces for the Hamiltonian Hénon mapping (14.22) with $a = 6$. Listed are the cycle itinerary, its expanding eigenvalue Λ_p, and its "center of mass". (The last one because we do not understand why the "center of mass" tends to be a simple rational every so often.)

we converge iteratively, at exponential rate, to the desired cycle points $x_{p,i}$. Given the cycle points, the cycle stabilities and periods are easily computed using (4.32). Verify that the times and the stabilities of the short periodic orbits for the Hénon repeller (3.10) at $a = 6$ are listed in table 14.2; in actual calculations all prime cycles up to topological length $n = 20$ have been computed.

(G. Vattay)

Exercise 14.12 "Center of mass" puzzle**. *Why is the "center of mass", listed in table 14.2, a simple rational every so often?*