Exercises

Exercise 20.1 **Diffusion for odd integer Λ.** *Show that when the slope $\Lambda = 2k - 1$ in (20.9) is an odd integer, the diffusion constant is given by $D = (\Lambda^2 - 1)/24$, as stated in (20.21).*

Exercise 20.2 **Fourth-order transport coefficient.** *Verify (20.24). You will need the identity*

$$\sum_{k=1}^{n} k^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1)\,.$$

Exercise 20.3 **Finite Markov partitions.** *Verify (20.28).*

Exercise 20.4 **Maps with variable peak shape:**
Consider the following piecewise linear map

$$f_\delta(x) = \begin{cases} \frac{3x}{1-\delta} & \text{for } x \in \left[0, \frac{1}{3}(1-\delta)\right] \\ \frac{3}{2} - \left(\frac{2}{\delta}\left|\frac{4-\delta}{12} - x\right|\right) & \text{for } x \in \left[\frac{1}{3}(1-\delta), \frac{1}{6}(2+\delta)\right] \\ 1 - \frac{3}{1-\delta}\left(x - \frac{1}{6}(2+\delta)\right) & \text{for } x \in \left[\frac{1}{6}(2+\delta), \frac{1}{2}\right] \end{cases} \qquad (20.38)$$

and the map in $[1/2, 1]$ is obtained by antisymmetry with respect to $x = 1/2$, $y = 1/2$. Write the corresponding dynamical zeta function relevant to diffusion and then show that

$$D = \frac{\delta(2+\delta)}{4(1-\delta)}$$

See refs. [20.17, 20.18] for further details.

Exercise 20.5 **Two-symbol cycles for the Lorentz gas.** *Write down all cycles labelled by two symbols, such as $(0\ 6)$, $(1\ 7)$, $(1\ 5)$ and $(0\ 5)$.*

Appendix P contains several project-length deterministic diffusion exercises.

Exercise 20.6 **Accelerated diffusion.** *Consider a map h, such that $\hat{h} = \hat{f}$ of fig. 20.6(b), but now running branches are turner into standing branches and vice versa, so that $1, 2, 3, 4$ are standing while 0 leads to both positive and negative jumps. Build the corresponding dynamical zeta function and show that*

$$\sigma^2(t) \sim \begin{cases} t & \textit{for } \alpha > 2 \\ t \ln t & \textit{for } \alpha = 2 \\ t^{3-\alpha} & \textit{for } \alpha \in (1,2) \\ t^2/\ln t & \textit{for } \alpha = 1 \\ t^2 & \textit{for } \alpha \in (0,1) \end{cases}$$

Exercise 20.7 Recurrence times for Lorentz gas with infinite horizon. *Consider the Lorentz gas with unbounded horizon with a square lattice geometry, with disk radius R and unit lattice spacing. Label disks according to the (integer) coordinates of their center: the sequence of recurrence times $\{t_j\}$ is given by the set of collision times. Consider orbits that leave the disk sitting at the origin and hit a disk far away after a free flight (along the horizontal corridor). Initial conditions are characterized by coordinates (ϕ, α) (ϕ determines the initial position along the disk, while α gives the angle of the initial velocity with respect to the outward normal: the appropriate measure is then $d\phi \cos\alpha \, d\alpha$ ($\phi \in [0, 2\pi)$, $\alpha \in [-\pi/2, \pi/2]$. Find how $\psi(T)$ scales for large values of T: this is equivalent to investigating the scaling of portions of the phase space that lead to a first collision with disk $(n, 1)$, for large values of n (as $n \mapsto \infty$ $n \simeq T$).*

Suggested steps

(a) *Show that the condition assuring that a trajectory indexed by (ϕ, α) hits the (m, n) disk (all other disks being transparent) is written as*

$$\left| \frac{d_{m,n}}{R} \sin\left(\phi - \alpha - \theta_{m,n}\right) + \sin\alpha \right| \leq 1 \qquad (20.39)$$

where $d_{m,n} = \sqrt{m^2 + n^2}$ and $\theta_{m,n} = \arctan(n/m)$. You can then use a small R expansion of (20.39).

(b) *Now call j_n the portion of the phase space leading to a first collision with disk $(n, 1)$ (take into account screening by disks $(1, 0)$ or $(n-1, 1)$). Denote by $J_n = \bigcup_{k=n+1}^{\infty} j_k$ and show that $J_n \sim 1/n^2$, from which the result for the distribution function follows.*