Exercises

Exercise 2.1 Trajectories do not intersect. *A trajectory in the phase space $\mathcal{M}$ is the set of points one gets by evolving $x \in \mathcal{M}$ forwards and backwards in time:*

$$C_x = \{y \in \mathcal{M} : f^t(x) = y \text{ for } t \in \mathbb{R}\} .$$

Show that if two trajectories intersect, then they are the same curve.

Exercise 2.2 Evolution as a group. *The trajectory evolution f^t is a one-parameter group where*

$$f^{t+s} = f^t \circ f^s .$$

Show that it is a commutative group.

In this case, the commutative character of the group of evolution functions comes from the commutative character of the time parameter under addition. Can you see any other group replacing time?

Exercise 2.3 Almost ODE's.

(a) *Consider the point x on $\mathbb{R}$ evolving according $\dot{x} = e^{\dot{x}}$. Is this an ordinary differential equation?*

(b) *Is $\dot{x} = x(x(t))$ an ordinary differential equation?*

(c) *What about $\dot{x} = x(t+1)$?*

Exercise 2.4 All equilibrium points are fixed points. *Show that a point of a vector field v where the velocity is zero is a fixed point of the dynamics f^t.*

Exercise 2.5 Gradient systems. *Gradient systems are a simple dynamical systems where the velocity field is given by the gradient of an auxiliary function ϕ*

$$\dot{x} = -\nabla\phi(x) .$$

x is a vector in $\mathbb{R}^d$, and ϕ a function from that space to the reals $\mathbb{R}$.

(a) *Show that the velocity of the particle is in the direction of most rapid decrease of the function ϕ.*

(b) *Show that all extrema of ϕ are fixed points of the flow.*

(c) *Show that it takes an infinite amount of time for the system to reach an equilibrium point.*

(d) *Show that there are no periodic orbits in gradient systems.*

Exercise 2.6 Runge-Kutta integration. *Implement the fourth-order Runge-Kutta integration formula (see, for example, ref. [2.9]) for $\dot{x} = v(x)$:*

$$\begin{aligned}
x_{n+1} &= x_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(\delta\tau^5) \\
k_1 &= \delta\tau v(x_n) , \qquad k_2 = \delta\tau v(x_n + k_1/2) \\
k_3 &= \delta\tau v(x_n + k_2/2) , \quad k_4 = \delta\tau v(x_n + k_3)
\end{aligned} \tag{2.18}$$

or some other numerical integration routine.

Exercise 2.7 Rössler system. *Use the result of exercise 2.6 or some other integration routine to integrate numerically the Rössler system (2.11). Does the result look like a "strange attractor"?*

Exercise 2.8 Can you integrate me? *Integrating equations numerically is not for the faint of heart. It is not always possible to establish that a set of nonlinear ordinary differential equations has a solution for all times and there are many cases were the solution only exists for a limited time interval, as, for example, for the equation $\dot{x} = x^2 , \quad x(0) = 1$.*

(a) *For what times do solutions of*

$$\dot{x} = x(x(t))$$

exist? Do you need numerical routine to answer this question?

(b) *Let's test the integrator you wrote in exercise 2.6. The equation $\ddot{x} = -x$ with initial conditions $x(0) = 2$ and $\dot{x} = 0$ has as solution $x(t) = e^{-t}(1 + e^{2\,t})$. Can your integrator reproduce this solution for the interval $t \in [0, 10]$? Check you solution by plotting the error as compared to the exact result.*

(c) *Now we will try something a little harder. The equation is going to be third order*

$$\dddot{x} + 0.6\ddot{x} + \dot{x} - |x| + 1 = 0 ,$$

which can be checked - numerically - to be chaotic. As initial conditions we will always use $\ddot{x}(0) = \dot{x}(0) = x(0) = 0$. Can you reproduce the result $x(12) = 0.8462071873$ (all digits are significant)? Even though the equation being integrated is chaotic, the time intervals are not long enough for the exponential separation of trajectories to be noticeble (the exponential growth factor is ≈ 2.4).

(d) *Determine the time interval for which the solution of $\dot{x} = x^2, x(0) = 1$ exists.*

Exercise 2.9 Classical collinear helium dynamics. *In order to apply the periodic orbit theory to quantization of helium we shall need to compute* *classical periodic orbits of the helium system. In this exercise we commence their evaluation for the collinear helium atom (5.3)*

$$H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - \frac{Z}{r_1} - \frac{Z}{r_2} + \frac{1}{r_1 + r_2}\,.$$

The nuclear charge for helium is $Z = 2$. *The colinear helium has only 3 degrees of freedom and the dynamics can be visualized as a motion in the* (r_1, r_2), $r_i \geq 0$ *quadrant. In the* (r_1, r_2) *coordinates the potential is singular for* $r_i \to 0$ *nucleus-electron collisions. These 2-body collisions can be regularized by rescaling the coordinates, with details given in sect. 29.1. In the transformed coordinates* (x_1, x_2, p_1, p_2) *the Hamiltonian equations of motion take the form (6.9).*

(a) *Integrate the equations of motion by the fourth order Runge-Kutta computer routine of exercise 2.6 (or whatever integration routine you like). A convenient way to visualize the 3-d phase space orbit is by projecting it onto the 2-dimensional* $(r_1(t), r_2(t))$ *plane.*

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Exercise 2.10 Infinite dimensional dynamical systems are not smooth. *Many of the operations we consider natural for finite dimensional systems do not have not smooth behavior in infinite dimensional vector spaces. Consider, as an example, a concentration* ϕ *diffusing on* $\mathbb{R}$ *according to the diffusion equation*

$$\partial_t \phi = \frac{1}{2}\nabla^2 \phi\,.$$

(a) *Interpret the partial differential equation as an infinite dimensional dynamical system. That is, write it as* $\dot{x} = F(x)$ *and find the velocity field.*

(b) *Show by examining the norm*

$$\|\phi\|^2 = \int_{\mathbb{R}} dx\, \phi^2(x)$$

that the vector field F *is not continuous.*

(c) *Try the norm*

$$\|\phi\| = \sup_{x\in\mathbb{R}} |\phi(x)|\,.$$

Is F *continuous?*

(d) *Argue that the semi-flow nature of the problem is not the cause of our difficulties.*

(e) *Do you see a way of generalizing these results?*