Exercises

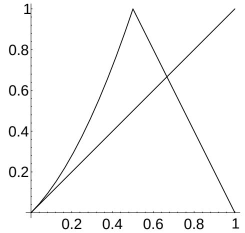
Exercise 18.1 Integral representation of Jonquière functions. *Check the integral representation*

$$J(z,\alpha) = \frac{z}{\Gamma(\alpha)} \int_0^\infty d\xi \, \frac{\xi^{\alpha-1}}{e^\xi - z} \qquad \text{for} \quad \alpha > 0\,. \tag{18.46}$$

Note how the denominator is connected to Bose-Einstein distribution. Compute $J(x+i\epsilon) - J(x-i\epsilon)$ for a real $x > 1$.

Exercise 18.2 Power law correction to a power law. *Expand (18.20) further and derive the leading power law correction to (18.23).*

Exercise 18.3 Power-law fall off. *In cycle expansions the stabilities of orbits do not always behave in a geometric fashion. Consider the map f*



This map behaves as $f \to x$ as $x \to 0$. Define a symbolic dynamics for this map by assigning 0 to the points that land on the interval $[0, 1/2)$ and 1 to the points that land on $(1/2, 1]$. Show that the stability of orbits that spend a long time on the 0 side goes as n^2. In particular, show that

$$\Lambda_{\underbrace{00\cdots0}_{n}1} \sim n^2$$

Exercise 18.4 Power law fall-off of stability eigenvalues in the stadium billiard.** *From the cycle expansions point of view, the most important consequence of the shear in $\mathbf{J}^n$ for long sequences of rotation bounces n_k in (5.23) is that the Λ_n grows only as a power law in number of bounces:*

$$\Lambda_n \propto n_k^2\,. \tag{18.47}$$

Check.

Exercise 18.5 Probabilistic zeta function for maps. *Derive the probabilistic zeta function for a map with recurrence distribution ψ_n.*

Exercise 18.6 Accelerated diffusion. *Consider a map h, such that $\hat{h} = \hat{f}$, but now running branches are turner into standing branches and vice versa, so that $1, 2, 3, 4$ are standing while 0 leads to both positive and negative jumps. Build the corresponding dynamical zeta function and show that*

$$\sigma^2(t) \sim \begin{cases} t & \text{for } \alpha > 2 \\ t \ln t & \text{for } \alpha = 2 \\ t^{3-\alpha} & \text{for } \alpha \in (1,2) \\ t^2/\ln t & \text{for } \alpha = 1 \\ t^2 & \text{for } \alpha \in (0,1) \end{cases}$$

Exercise 18.7 Anomalous diffusion (hyperbolic maps). *Anomalous diffusive properties are associated to deviations from linearity of the variance of the phase variable we are looking at: this means the the diffusion constant (8.13) either vanishes or diverges. We briefly illustrate in this exercise how the local local properties of a map are crucial to account for anomalous behavior even for hyperbolic systems.*

Consider a class of piecewise linear maps, relevant to the problem of the onset of diffusion, defined by

$$f_\epsilon(x) = \begin{cases} \Lambda x & \text{for } x \in \left[0, x_1^+\right] \\ a - \Lambda_{\epsilon,\gamma}|x - x^+| & \text{for } x \in \left[x_1^+, x_2^+\right] \\ 1 - \Lambda'(x - x_2^+) & \text{for } x \in \left[x_2^+, x_1^-\right] \\ 1 - a + \Lambda_{\epsilon,\gamma}|x - x^-| & \text{for } x \in \left[x_1^-, x_2^-\right] \\ 1 + \Lambda(x-1) & \text{for } x \in \left[x_2^-, 1\right] \end{cases} \tag{18.48}$$

where $\Lambda = (1/3 - \epsilon^{1/\gamma})^{-1}$, $\Lambda' = (1/3 - 2\epsilon^{1/\gamma})$, $\Lambda_{\epsilon,\gamma} = \epsilon^{1-1/\gamma}$, $a = 1+\epsilon$, $x^+ = 1/3$, $x_1^+ = x^+ - \epsilon^{1/\gamma}$, $x_2^+ = x^+ + \epsilon^{1/\gamma}$, and the usual symmetry properties (20.11) are satisfied.

Thus this class of maps is characterized by two escaping windows (through which the diffusion process may take place) of size $2\epsilon^{1/\gamma}$: the exponent γ mimicks the order of the maximum for a continuous map, while piecewise linearity, besides making curvatures vanish and leading to finite cycle expansions, prevents the appearance of stable cycles. The symbolic dynamics is easily described once we consider a sequence of parameter values $\{\epsilon_m\}$, where $\epsilon_m = \Lambda^{-(m+1)}$: we then partition the unit interval though the sequence of points $0, x_1^+, x^+, x_2^+, x_1^-, x^-, x_2^-, 1$ and label the corresponding sub–intervals $1, s_a, s_b, 2, d_b, d_a, 3$: symbolic dynamics is described by an unrestricted grammar over the following set of symbols

$$\{1, 2, 3, s_\# \cdot 1^i, d_\# \cdot 3^k\} \qquad \# = a, b \quad i, k = m, m+1, m+2, \ldots$$

This leads to the following dynamical zeta function:

$$\zeta_0^{-1}(z,\alpha) = 1 - \frac{2z}{\Lambda} - \frac{z}{\Lambda'} - 4\cosh(\alpha)\epsilon_m^{1/\gamma - 1} \frac{z^{m+1}}{\Lambda^m}\left(1 - \frac{z}{\Lambda}\right)^{-1}$$

from which, by (20.8) we get

$$D = \frac{2\epsilon_m^{1/\gamma-1}\Lambda^{-m}(1-1/\Lambda)^{-1}}{1-\frac{2}{\Lambda}-\frac{1}{\Lambda'}-4\epsilon_m^{1/\gamma-1}\left(\frac{m+1}{\Lambda^m(1-1/\Lambda)}+\frac{1}{\Lambda^{m+1}(1-1/\Lambda)^2}\right)} \tag{18.49}$$

The main interest in this expression is that it allows exploring how D vanishes in the $\epsilon \mapsto 0$ ($m \mapsto \infty$) limit: as a matter of fact, from (18.49) we get the asymptotic behavior $D \sim \epsilon^{1/\gamma}$, which shows how the onset of diffusion is governed by the order of the map at its maximum.

Remark 18.5 Onset of diffusion for continuous maps. The zoology of behavior for continuous maps at the onset of diffusion is described in refs. [20.11, 20.12, 18.24]: our treatment for piecewise linear maps was introduced in ref. [18.25].