Exercises

Exercise 21.1 Mode-locked intervals. *Check that when $k \neq 0$ the interval $\Delta_{P/Q}$ have a non-zero width (look for instance at simple fractions, and consider k small). Show that for small k the width of $\Delta_{0/1}$ is an increasing function of k.*

Exercise 21.2 Bounds on Hausdorff dimension. *By making use of the bounds (21.17) show that the Hausdorff dimension for critical mode lockings may be bounded by*

$$2/3 \leq D_H \leq .9240\ldots$$

Exercise 21.3 Farey model sum rules. *Verify the sum rules reported in table 21.2. An elegant way to get a number of sum rules for the Farey model is by taking into account an lexical ordering introduced by Contucci and Knauf, see ref. [21.28].*

Exercise 21.4 Metric entropy of the Gauss shift. *Check that the Lyapunov exponent of the Gauss map (21.7) is given by $\pi^2/6\ln 2$. This result has been claimed to be relevant in the discussion of "mixmaster" cosmologies, see ref. [21.30].*

Exercise 21.5 Refined expansions. *Show that the above estimates can be refined as follows:*

$$F(z,2) \sim \zeta(2) + (1-z)\log(1-z) - (1-z)$$

and

$$F(z,s) \sim \zeta(s) + \Gamma(1-s)(1-z)^{s-1} - S(s)(1-z)$$

for $s \in (1,2)$ ($S(s)$ being expressed by a converging sum). You may use either more detailed estimate for $\zeta(s,a)$ (via Euler summation formula) or keep on subtracting leading contributions [21.31].

Exercise 21.6 Hitting condition. *Prove (20.39). Hint: together with the real trajectory consider the line passing through the starting point, with polar angle $\theta_{m,n}$: then draw the perpendiculars to the actual trajectory, passing through the center of the $(0,0)$ and (m,n) disks.*

Exercise 21.7 j_n and α_{cr}. *Look at the integration region and how it scales by plotting it for increasing values of n.*

Exercise 21.8 Estimates of the Riemann zeta function. *Try to approximate numerically the Riemann zeta function for $s = 2,4,6$ using different acceleration algorithms: check your results with refs. [21.32, 21.33].*

Exercise 21.9 Farey tree and continued fractions I. *Consider the Farey tree presentation function* $f : [0,1] \mapsto [0,1]$*, such that if* $I = [0, 1/2)$ *and* $J = [1/2, 1]$*,* $f|_I = x/(1-x)$ *and* $f|_J = (1-x)/x$*. Show that the corresponding induced map is the Gauss map* $g(x) = 1/x - [1/x]$*.*

Exercise 21.10 Farey tree and continued fraction II. (Lethal weapon II). *Build the simplest piecewise linear approximation to the Farey tree presentation function (hint: substitute first the righmost, hyperbolic branch with a linear one): consider then the spectral determinant of the induced map* $\hat{g}$*, and calculate the first two eigenvalues besides the probability conservation one. Compare the results with the rigorous bound deduced in ref. [18.17].*