Exercises

Exercise 15.1 Cycle expansions. *Write programs that implement* binary *symbolic dynamics cycle expansions for (a) dynamical zeta functions, (b) spectral determinants. Combined with the cycles computed for a 2-branch repeller or a 3-disk system they will be useful in problem that follow.*

Exercise 15.2 Escape rate for a 1-d repeller. *(Continuation of exercise 12.1 - easy, but long)*
Consider again the quadratic map (12.31)

$$f(x) = Ax(1-x)$$

on the unit interval, for definitivness take either $A = 9/2$ or $A = 6$. Describing the itinerary of any trajectory by the binary alphabet $\{0, 1\}$ ('0' if the iterate is in the first half of the interval and '1' if is in the second half), we have a repeller with a complete binary symbolic dynamics.

(a) *Sketch the graph of f and determine its two fixed points $\overline{0}$ and $\overline{1}$, together with their stabilities.*

(b) *Sketch the two branches of f^{-1}. Determine all the prime cycles up to topological length 4 using your pocket calculator and backwards iteration of f (see sect. 14.2.1).*

(c) *Determine the leading zero of the zeta function (12.12) using the weigths $t_p = z^{n_p}/|\Lambda_p|$ where Λ_p is the stability of the p cycle.*

(d) *Show that for $A = 9/2$ the escape rate of the repeller is $0.361509\ldots$ using the spectral determinant, with the same cycle weight. If you have taken $A = 6$, the escape rate is in $0.83149298\ldots$, as shown in solution 15.2. Compare the coefficients of the spectral determinant and the zeta function cycle expansions. Which expansion converges faster?*

(Per Rosenqvist)

Exercise 15.3 Escape rate for the Ulam map. *Check that the escape rate for the Ulam map, $A = 4$ in (12.31)*

$$f(x) = 4x(1-x),$$

equals zero. You might note that the convergence as function of the truncation cycle length is slow. Try to fix that by treating the $\Lambda_0 = 4$ cycle separately.

Exercise 15.4 **Pinball escape rate, semi-analytical.** *Estimate the 3-disk pinball escape rate for $R : a = 6$ by substituting analytical cycle stabilities and periods (exercise 5.5 and exercise 5.6) into the appropriate binary cycle expansion. Compare with the numerical estimate exercise 12.12*

Exercise 15.5 **Pinball escape rate, from numerical cycles.** *Compute the escape rate for $R : a = 6$ 3-disk pinball by substituting list of numerically computed cycle stabilities of exercise 14.5 into the binary cycle expansion.*

Exercise 15.6 **Pinball resonances, in the complex plane.** *Plot the logarithm of the absolute value of the dynamical zeta function and/or the spectral determinant cycle expansion (15.5) as contour plots in the complex s plane. Do you find zeros other than the one corresponding to the complex one? Do you see evidence for a finite radius of convergence for either cycle expansion?*

Exercise 15.7 **Counting the 3-disk pinball counterterms.** *Verify that the number of terms in the 3-disk pinball curvature expansion (15.31) is given by*

$$\begin{aligned}\prod_p (1+t_p) &= \frac{1-3z^4-2z^6}{1-3z^2-2z^3} = 1+3z^2+2z^3+\frac{z^4(6+12z+2z^2)}{1-3z^2-2z^3} \\ &= 1+3z^2+2z^3+6z^4+12z^5+20z^6+48z^7+84z^8+184z^9+\ldots\end{aligned}$$

This means that, for example, c_6 has a total of 20 terms, in agreement with the explicit 3-disk cycle expansion (15.32).

Exercise 15.8 **3–disk unfactorized zeta cycle expansions.** *Check that the curvature expansion (15.2) for the 3-disk pinball, assuming no symmetries between disks, is given by*

$$\begin{aligned}1/\zeta &= (1-z^2t_{12})(1-z^2t_{13})(1-z^2t_{23})(1-z^3t_{123})(1-z^3t_{132}) \\ &\quad (1-z^4t_{1213})(1-z^4t_{1232})(1-z^4t_{1323})(1-z^5t_{12123})\cdots \\ &= 1-z^2t_{12}-z^2t_{23}-z^2t_{31}-z^3t_{123}-z^3t_{132} \\ &\quad -z^4[(t_{1213}-t_{12}t_{13})+(t_{1232}-t_{12}t_{23})+(t_{1323}-t_{13}t_{23})] \\ &\quad -z^5[(t_{12123}-t_{12}t_{123})+\cdots]-\cdots \end{aligned} \tag{15.31}$$

The symmetrically arranged 3-disk pinball cycle expansion of the Euler product (15.2) (see table 10.4 and fig. 19.2) is given by:

$$\begin{aligned}1/\zeta &= (1-z^2t_{12})^3(1-z^3t_{123})^2(1-z^4t_{1213})^3 \\ &\quad (1-z^5t_{12123})^6(1-z^6t_{121213})^6(1-z^6t_{121323})^3\ldots \\ &= 1-3z^2\,t_{12}-2z^3\,t_{123}-3z^4\,(t_{1213}-t_{12}^2)-6z^5\,(t_{12123}-t_{12}t_{123}) \\ &\quad -z^6\,(6\,t_{121213}+3\,t_{121323}+t_{12}^3-9\,t_{12}t_{1213}-t_{123}^2) \\ &\quad -6z^7\,(t_{1212123}+t_{1212313}+t_{1213123}+t_{12}^2t_{123}-3\,t_{12}t_{12123}-t_{123}t_{1213}) \\ &\quad -3z^8\,(2\,t_{12121213}+t_{12121313}+2\,t_{12121323}+2\,t_{12123123} \\ &\qquad +2\,t_{12123213}+t_{12132123}+3\,t_{12}^2t_{1213}+t_{12}t_{123}^2 \\ &\qquad -6\,t_{12}t_{121213}-3\,t_{12}t_{121323}-4\,t_{123}t_{12123}-t_{1213}^2)-\cdots \end{aligned} \tag{15.32}$$

Remark 15.7 Unsymmetrized cycle expansions. The above 3-disk cycle expansions might be useful for cross-checking purposes, but, as we shall see in chapter 19, they are not recommended for actual computations, as the factorized zeta functions yield much better convergence.

Exercise 15.9 **4–disk unfactorized dynamical zeta function cycle expansions** *For the symmetriclly arranged 4-disk pinball the symmetry group is C_{4v}, of order 8. The degenerate cycles can have multiplicities 2, 4 or 8 (see table 10.2):*

$$\begin{aligned} 1/\zeta \;=\; & (1-z^2t_{12})^4(1-z^2t_{13})^2(1-z^3t_{123})^8(1-z^4t_{1213})^8(1-z^4t_{1214})^4 \\ & (1-z^4t_{1234})^2(1-z^4t_{1243})^4(1-z^5t_{12123})^8(1-z^5t_{12124})^8(1-z^5t_{12134})^8 \\ & (1-z^5t_{12143})^8(1-z^5t_{12313})^8(1-z^5t_{12413})^8\cdots \end{aligned} \tag{15.33}$$

and the cycle expansion is given by

$$\begin{aligned} 1/\zeta \;=\; & 1-z^2(4\,t_{12}+2\,t_{13})-8z^3\,t_{123} \\ & -z^4(8\,t_{1213}+4\,t_{1214}+2\,t_{1234}+4\,t_{1243}-6\,t_{12}^2-t_{13}^2-8\,t_{12}t_{13}) \\ & -8z^5(t_{12123}+t_{12124}+t_{12134}+t_{12143}+t_{12313}+t_{12413}-4\,t_{12}t_{123}-2\,t_{13}t_{123}) \\ & -4z^6(2\,S_8+\,S_4+\,t_{12}^3+3\,t_{12}^2\,t_{13}+\,t_{12}t_{13}^2-8\,t_{12}t_{1213}-4\,t_{12}t_{1214} \\ & -2\,t_{12}t_{1234}-4\,t_{12}t_{1243}-4\,t_{13}t_{1213}-2\,t_{13}t_{1214}-\,t_{13}t_{1234} \\ & -2\,t_{13}t_{1243}-7\,t_{123}^2)-\cdots \end{aligned} \tag{15.34}$$

where in the coefficient to z^6 the abbreviations S_8 and S_4 stand for the sums over the weights of the 12 orbits with multiplicity 8 and the 5 orbits of multiplicity 4, respectively; the orbits are listed in table 10.4.

Exercise 15.10 **Tail resummations.** *A simple illustration of such tail resummation is the ζ function for the Ulam map (14.21) for which the cycle structure is exceptionally simple: the eigenvalue of the $x_0=0$ fixed point is 4, while the eigenvalue of any other n-cycle is $\pm 2^n$. Typical cycle weights used in thermodynamic averaging are $t_0=4^\tau z$, $t_1=t=2^\tau z$, $t_p=t^{n_p}$ for $p\neq 0$. The simplicity of the cycle eigenvalues enables us to evaluate the ζ function by a simple trick: we note that if the value of any n-cycle eigenvalue were t^n, (12.18) would yield $1/\zeta=1-2t$. There is only one cycle, the x_0 fixed point, that has a different weight $(1-t_0)$, so we factor it out, multiply the rest by $(1-t)/(1-t)$, and obtain a rational ζ function*

$$1/\zeta(z) \;=\; \frac{(1-2t)(1-t_0)}{(1-t)} \tag{15.35}$$

Consider how we would have detected the pole at $z=1/t$ without the above trick. As the $\overline{0}$ fixed point is isolated in its stability, we would have kept the factor $(1-t_0)$ in (15.5) unexpanded, and noted that all curvature combinations in (15.5) which include the t_0 factor are unbalanced, so that the cycle expansion is an infinite series:

$$\prod_p (1-t_p) \;=\; (1-t_0)(1-t-t^2-t^3-t^4-\ldots) \tag{15.36}$$

(we shall return to such infinite series in chapter 18). The geometric series in the brackets sums up to (15.35). Had we expanded the $(1 - t_0)$ *factor, we would have noted that the ratio of the successive curvatures is exactly* $c_{n+1}/c_n = t$*; summing we would recover the rational* ζ *function (15.35).*