Exercises

Exercise L.1 **Not all Banach spaces are also Hilbert** *If we are given a norm $\|\cdot\|$ of a Banach space B, it may be possible to find an inner product $\langle\cdot\,,\cdot\rangle$ (so that B is also a Hilbert space H) such that for all vectors $f \in B$, we have*

$$\|f\| = \langle f, f\rangle^{1/2} .$$

This is the norm induced by the scalar product. If we cannot find the inner product how do we know that we just are not being clever enough? By checking the parallelogram law for the norm. A Banach space can be made into a Hilbert space if and only if the norm satisfies the parallelogram law. The parallelogram law says that for any two vectors f and g the equality

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2 ,$$

must hold.

Consider the space of bounded observables with the norm given by $\|a\| = \sup_{\sigma\in\Omega^{\mathbb{N}}} |a(\sigma)|$. Show that ther is no scalar product that will induce this norm.

Exercise L.2 **Automaton for a droplet** *Find the Markov graph and the weights on the edges so that the energies of configurations for the dropolet model are correctly generated. For any string starting in zero and ending in zero your diagram should yield a configuration the weight $e^{H(\sigma)}$, with H computed along the lines of (L.13) and (L.18).*

Hint: the Markov graph is infinite.

Exercise L.3 **Spectral determinant for a^n interactions** *Compute the spectral determinant for one-dimensional Ising model with the interaction*

$$\phi(\sigma) = \sum_{k>0} a^k \delta(\sigma_0, \sigma_k) .$$

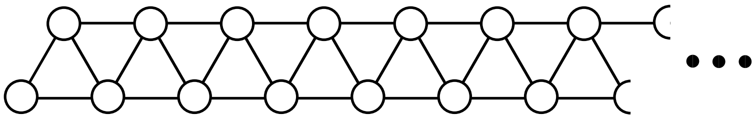
Take a as a number smaller than $1/2$.

(a) *What is the dynamical system this generates? That is, find F_+ and F_- as used in (L.39).*

(b) *Show that*

$$\frac{d}{dx}F_{\{+\text{ or}-\}} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

Exercise L.4 **Ising model on a thin strip** *Compute the transfer matrix for the Ising model defined on the graph*



Assume that whenever there is a bond connecting two sites, there is a contribution $J\delta(\sigma_i, \sigma_j)$ to the energy.

Exercise L.5 **Infinite symbolic dynamics** *Let σ be a function that returns zeo or one for every infinite binary string: $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}$. Its value is represented by $\sigma(\epsilon_1, \epsilon_2, \ldots)$ where the ϵ_i are either 0 or 1. We will now define an operator $\mathcal{T}$ that acts on observables on the space of binary strings. A function a is an observable if it has bounded variation, that is, if*

$$\|a\| = \sup_{\{\epsilon_i\}} |a(\epsilon_1, \epsilon_2, \ldots)| < \infty \,.$$

For these functions

$$\mathcal{T}a(\epsilon_1, \epsilon_2, \ldots) = a(0, \epsilon_1, \epsilon_2, \ldots)\sigma(0, \epsilon_1, \epsilon_2, \ldots) + a(1, \epsilon_1, \epsilon_2, \ldots)\sigma(1, \epsilon_1, \epsilon_2, \ldots) \,.$$

The function σ is assumed such that any of $\mathcal{T}$'s "matrix representations" in (a) have the Markov property (the matrix, if read as an adjacency graph, corresponds to a graph where one can go from any node to any other node).

(a) *(easy) Consider a finite version T_n of the operator $\mathcal{T}$:*

$$\begin{aligned} &T_n a(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) = \\ &\qquad a(0, \epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1})\sigma(0, \epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1}) + \\ &\qquad a(1, \epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1})\sigma(1, \epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1}) \,. \end{aligned}$$

Show that T_n is a $2^n \times 2^n$ matrix. Show that its trace is bounded by a number independent of n.

(b) *(medium) With the operator norm induced by the function norm, show that $\mathcal{T}$ is a bounded operator.*

(c) *(hard) Show that $\mathcal{T}$ is not trace-class. (Hint: check if $\mathcal{T}$ is compact).*

Classes of operators are nested; trace-class $\leq$ compact $\leq$ bounded.