Exercises

Exercise 24.1 WKB ansatz. *Try to show that no other ansatz other than the (26.1) gives a meaningful definition of the momentum in the $\hbar \to 0$ limit.*

Exercise 24.2 1-dimensional harmonic oscillator. *Take a 1-dimensional harmonic oscillator $U(q) = \frac{1}{2}kq^2$. Take a WKB wave function of form $A(q,t) = a(t)$ and $R(q,t) = r(t) + b(t)q + c(t)q^2$, where $r(t), a(t), b(t)$ and $c(t)$ are time dependent coefficients. Derive ordinary differential equations by using (26.3) and (26.4) and solve them. Continuation: (24.5).*

Exercise 24.3 1-dimensional linear potential. *Take a 1-dimensional linear potential $U(q) = -Fq$. Take a WKB wave function of form $A(q,t) = a(t)$ and $R(q,t) = r(t) + b(t)q + c(t)q^2$, where $r(t), a(t), b(t)$ and $c(t)$ are time dependent coefficients. Derive and solve the ordinary differential equations from (26.3) and (26.4).*

Exercise 24.4 d-dimensional quadratic potentials. *Generalize the above method to general d-dimensional quadratic potentials.*

Exercise 24.5 Time evolution of R. *(Continuation of (24.2)). Calculate the time evolution of $R(q,0) = a + bq + cq^2$ for a 1-dimensional harmonic oscillator using (26.12) and (26.13).*

Exercise 24.6 Fresnel integral. *Show that*

$$\int_{-\infty}^{\infty} dx\, e^{iax^2/2} = \left(\frac{2\pi}{|a|}\right)^{1/2} e^{\frac{i\pi m}{4}} \tag{24.22}$$

where $m = a/|a|$ is the sign of a.

Exercise 24.7 Stationary phase approximation. *All semiclassical approximations are based on saddlepoint evaluations of integrals of type*

$$I = \int d^d x A(x) e^{i\Phi(x)/\hbar} \tag{24.23}$$

for small values of $\hbar$. Obtain the stationary phase estimate

$$I \approx \sum_n A(x_n) e^{i\Phi(x_n)/\hbar} \frac{(2\pi i\hbar)^{d/2}}{\sqrt{\det \mathbf{D}^2\Phi(x_n)}},$$

where $\mathbf{D}^2\Phi(x_n)$ denotes the second derivative matrix.

Exercise 24.8 Sterling formula for $n!$. *Compute an approximate value of $n!$ for large n with the help of stationary phase approximation. Hint: $n! = \int_0^\infty dt\, t^n e^{-t}$.*

Exercise 24.9 Airy function for large arguments. *Important contributions as stationary phase points may arise from extremal points where the first non-zero term in a Taylor expansion of the phase is of third or higher order. Such situations occur, for example, at bifurcation points or in diffraction effects, (such as waves near sharp corners, waves creeping around obstacles). In such calculations one meets Airy functions integrals of the form*

$$Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dy\, e^{i(xy - \frac{y^3}{3})} . \tag{24.24}$$

Calculate the Airy function $Ai(x)$ by stationary phase approximation. What happens when considering the limit $x \to 0$. Estimate for which value of x the stationary phase approximation breaks down. Give it a go.