Chapter 2

Flows

Poetry is what is lost in translation
Robert Frost

(R. Mainieri, P. Cvitanović and E.A. Spiegel)

We start out by a recapitulation of the basic notions of dynamics. Our aim is narrow; keep the exposition focused on prerequisites to the applications to be developed in this text. We assume that the reader is familiar with the dynamics on the level of introductory texts mentioned in sect. 1.8, and concentrate here on developing intuition about what a dynamical system can do. It will be a coarse brush sketch - a full description of all possible behaviors of dynamical systems is anyway beyond human ken. For a novice there is no shortcut through this lengthy detour; a sophisticated traveler might prefer to skip this well trodden territory and embark upon the journey at chapter 7.

fast track:
chapter 7, p. 105

2.1 Dynamical systems

In a dynamical system we observe the world as a function of time. We express our observations as numbers and record how they change with time; given sufficiently detailed information and understanding of the underlying natural laws, we see the future in the present as in a mirror. The motion of the planets against the celestial firmament provides an example. Against the daily motion of the stars from East to West, the planets distinguish themselves by moving among the fixed stars. Ancients discovered that by

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knowing a sequence of planet's positions - latitudes and longitudes - its future position could be predicted.

For the solar system, tracking the latitude and longitude in the celestial sphere suffices to completely specify the planet's apparent motion. All possible values for positions and velocities of the planets form the *phase space* of the system. More generally, a state of a physical system, at a given instant in time, can be represented by a single point in an abstract space called *state space* or *phase space* $\mathcal{M}$. As the system changes, so does the *representative point* in phase space. We refer to the evolution of such points as *dynamics*, and the function f^t which specifies where the representative point is at time t as the *evolution rule*.

If there is a definite rule f that tells us how this representative point moves in $\mathcal{M}$, the system is said to be deterministic. For a deterministic dynamical system, the evolution rule takes one point of the phase space and maps it into another point. Not two or three, but exactly one. This is not always possible. For example, knowing the temperature today is not enough to predict the temperature tomorrow; or knowing the value of a stock market index today will not determine its value tomorrow. The phase space can be enlarged, in the hope that in a sufficiently large phase space it is possible to determine an evolution rule, so we imagine that knowing the state of the atmosphere, measured over many points over the entire planet should be sufficient to determine the temperature tomorrow. Even that is not quite true, and we are less hopeful when it comes to a stock index.

For a deterministic system almost every point has a unique future, so trajectories cannot intersect. We say "almost" because there might exist a set of measure zero (tips of wedges, cusps, *etc.*) for which a trajectory is not defined. We may think such sets a nuisance, but it is quite the contrary - they will enable us to partition phase space, so that the dynamics can be better understood.

Locally, the phase space $\mathcal{M}$ is $\mathbb{R}^d$, meaning that d numbers are sufficient to determine what will happen next. Globally, it may be a more complicated manifold formed by patching together several pieces of $\mathbb{R}^d$, forming a torus, a cylinder, or some other manifold. When we need to stress that the dimension d of $\mathcal{M}$ is greater than one, we may refer to the point $x \in \mathcal{M}$ as x_i where $i = 1, 2, 3, \ldots, d$. The *evolution rule* or *dynamics* $f^t : \mathcal{M} \to \mathcal{M}$ tells us where a point x is in $\mathcal{M}$ after a time interval t. The pair $(\mathcal{M}, f)$ is called a *dynamical system*.

The dynamical systems we will be studying are smooth. This is expressed mathematically, by saying that the evolution rule f^t can be differentiated as many times as needed. Its action on a point x is sometimes indicated by $f(t, x)$ to remind us that f is really a function of two variables: time interval and point of phase space. Note that time is not absolute, only

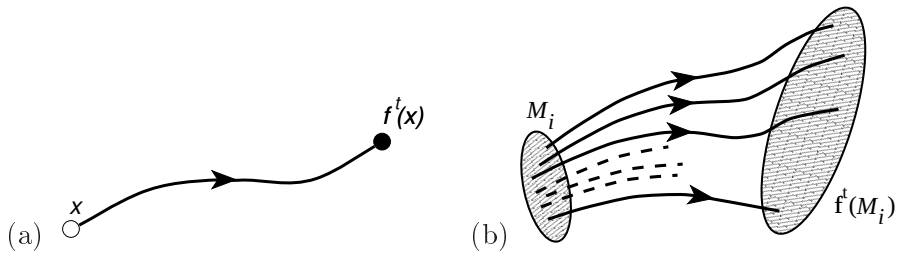

Figure 2.1: (a) A trajectory traced out by the evolution rule f^t. Starting from the phase space point x, after a time t, the point is at $f^t(x)$. (b) The evolution rule f^tcan be used to map a region $\mathcal{M}_i$ of the phase space into the region $f^t(\mathcal{M}_i)$.

the time interval is necessary. This is because a point in phase space completely determines all future evolution, and it is not necessary to know anything else. The time parameter can be a real variable ($t \in \mathbb{R}$), in which case the evolution is called a *flow*, or an integer ($t \in \mathbb{Z}$), in which case the evolution advances in discrete steps in time, given by *iteration* of a *map*.

Nature provides us with innumerable dynamical systems. They manifest themselves through their trajectories: given an initial point x_0, the evolution rule traces out a sequence of points $x(t) = f^t(x_0)$, the *trajectory* through the point $x_0 = x(0)$. A trajectory is parameterized by the time t and thus belongs to $(f^t(x_0), t) \in \mathcal{M} \otimes \mathbb{R}$. By extension, we can also talk of the evolution of a region $\mathcal{M}_i$ of the phase space: just apply f^t to every point in $\mathcal{M}_i$ to obtain a new region $f^t(\mathcal{M}_i)$, as in fig. 2.1.

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Because f^t is a single-valued function, any point of the trajectory can be used to label the trajectory. If we mark the trajectory by its inital point x_0, we are describing it in the *Lagrangian coordinates*. We can regard the transport of the material point at $t = 0$ to its current point $x(t) = f^t(x_0)$ as a coordinate transformation from the Lagrangian coordinates to the *Eulerian coordinates*.

The subset of points in $\mathcal{M}$ that belong to the (possibly infinite) trajectory of a given point x_0 is called the *orbit* of x_0; we shall talk about forward orbits, backward orbits, periodic orbits. For a flow, an orbit is a continuous curve; for a map, a sequence of points.

What are the possible trajectories? This is a grand question, and there are many answers, chapters to follow offering some. Here we shall classify possible trajectories as:

stationary:	$f^t(x) = x$	for all t
periodic:	$f^t(x) = f^{t+T_p}(x)$	for a given minimum period T_p
aperiodic:	$f^t(x) \neq f^{t'}(x)$	for all $t \neq t'$.

The ancient, no less than the contemporary quantum field theorists,

tried to make sense of all dynamics in terms of periodic motions; epicycles, integrable systems. The embarassing truth is that for a generic dynamical systems most motions are aperiodic. We will break aperiodic motions up into two types: those that wander off, and those that keep coming back.

A point $x \in \mathcal{M}$ is called a *wandering point*, if there exists an open neighborhood $\mathcal{M}_0$ of x to which the trajectory never returns

$$f^t(x) \notin \mathcal{M}_0 \qquad \text{for all} \quad t > t_{min}\,. \tag{2.1}$$

In physics literature, the dynamics of such state is often referred to as *transient*.

A *periodic orbit* corresponds to a trajectory that returns exactly to the initial point in a finite time. Periodic orbits form a very small subset of the phase space, in the same sense that rational numbers are a set of zero measure on the unit interval. For times much longer than a typical "turnover" time, it makes sense to relax the notion of exact periodicity, and replace it by the notion of *recurrence*. A point is *recurrent* or *non-wandering* if for any open neighborhood $\mathcal{M}_0$ of x and any time t_{min} there exists a later time t, such that

$$f^t(x) \in \mathcal{M}_0\,. \tag{2.2}$$

In other words, the trajectory of a non-wandering point reenters the neighborhood $\mathcal{M}_0$ infinitely often. We shall denote by Ω the *non–wandering set* of f, that is the union of all the non-wandering points of $\mathcal{M}$. The set Ω, the non–wandering set of f, is the key to understanding the long-time behavior of a dynamical system; all calculations undertaken here will be carried out on non–wandering sets.

So much about individual trajectories. What about clouds of initial points? If there exists a connected phase space volume that maps into itself under the forward evolution (and you can prove that by the method of Lyapunov functionals, or any other method available in the literature), the flow is globally contracting onto a subset of $\mathcal{M}$ which we shall refer to as the *attractor*. The attractor may be unique, or there can coexist any number of distinct attracting sets, each with its own *basin of attraction*, the set of all points that fall into the attractor under foward evolution. The attractor can be a fixed point, a periodic orbit, aperiodic, or any combination of the above. The most interesting case is that of an aperiodic reccurent attractor, to which we shall refer loosely as a *strange attractor*. We say loosely, as will soon become apparent, that diagnosing and proving existence of a genuine, card carrying strange attractor is a highly nontrivial undertaking.

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Conversely, if we can enclose the non–wandering set Ω by a connected phase space volume $\mathcal{M}_0$ and then show that almost all points within $\mathcal{M}_0$, but not in Ω, eventually exit $\mathcal{M}_0$, we refer to the non–wandering set Ω as a *repeller*. An example of a repeller is not hard to come by - the pinball game of sect. 1.3 is a simple chaotic repeller.

It would seem, that having said that the periodic points are too exceptional, and that almost all non-wandering points are aperiodic, we have given up the ancients' fixation on periodic motions. Quite to the contrary. As longer and longer cycles approximate more and more accurately finite segments of aperiodic trajectories, we shall establish control over non–wandering sets by defining them as the closures of the union of all periodic points.

Before we can work out an example of a non–wandering set and get a better grip on what chaotic motion might look like, we need to ponder flows in a little more depth.

2.2 Flows

> There is no beauty without some strangeness
> William Blake

A *flow* is a continuous-time dynamical system. The evolution rule f^t is a family of mappings of $\mathcal{M} \to \mathcal{M}$ parameterized by $t \in \mathbb{R}$. Because t represents a time interval, any family of mappings that forms an evolution rule must satisfy:

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(a) $f^0(x) = x$ (in 0 time there is no motion)

(b) $f^t(f^{t'}(x)) = f^{t+t'}(x)$ (the evolution law is the same at all times)

(c) the mapping $(x,t) \mapsto f^t(x)$ from $\mathcal{M} \times \mathbb{R}$ into $\mathcal{M}$ is continuous.

The family of mappings $f^t(x)$ thus forms a continuous (forward semi-) group. Why "semi-"group? It may fail to form a group if the dynamics is not reversible, and the rule $f^t(x)$ cannot be used to rerun the dynamics backwards in time, with negative t; with no reversibility, we cannot define the inverse $f^{-t}(f^t(x)) = f^0(x) = x$, in which case the family of mappings $f^t(x)$ does not form a group. In exceedingly many situations of interest - for times beyond the Lyapunov time, for asymptotic attractors, for infinite dimensional systems, for systems with noise, for non-invertible maps - the dynamics cannot be run backwards in time, hence, the circumspect

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emphasis on *semi*groups. On the other hand, there are many settings of physical interest, where dynamics is reversible (such as finite-dimensional Hamiltonian flows), and where the family of evolution maps f^t does form a group.

For infinitesimal times, flows can be defined by differential equations. Write a trajectory as

$$x(t+\tau) = f^{t+\tau}(x_0) = f(f(x_0,t),\tau) \tag{2.3}$$

and express the time derivative of a trajectory at point $x(t)$,

$$\left.\frac{dx}{d\tau}\right|_{\tau=0} = \partial_\tau f(f(x_0,t),\tau)|_{\tau=0} = \dot{x}(t)\,. \tag{2.4}$$

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as the time derivative of the evolution rule, a vector evaluated at the same point. By considering all possible trajectories, we obtain the vector $\dot{x}(t)$ at any point $x \in \mathcal{M}$; this *vector field* is a (generalized) velocity field:

$$v(x) = \dot{x}(t)\,. \tag{2.5}$$

Newton's laws, Lagrange's method, or Hamilton's method are all familiar procedures for obtaining a set of differential equations for the vector field $v(x)$ that describes the evolution of a mechanical system. Equations of mechanics may appear different in form from (2.5), as they are often involve higher time derivatives, but an equation that is second or higher order in time can always be rewritten as a set of first order equations.

We are concerned here with a much larger world of general flows, mechanical or not, all defined by a time independent vector field (2.5). At each point of the phase space a vector indicates the local direction in which the orbit evolves. The length of the vector $|v(x)|$ is proportional to the speed at the point x, and the direction and length of $v(x)$ changes from point to point. When the phase space is a complicated manifold embedded in $\mathbb{R}^d$, one can no longer think of the vector field as being embedded in the phase space. Instead, we have to imagine that each point x of phase space has a different tangent plane $T\mathcal{M}_x$ attached to it. The vector field lives in the union of all these tangent planes, a space called the *tangent bundle* $\mathbf{T}\mathcal{M}$.

Example 2.1 A two-dimensional vector field $v(x)$**:** *A simple example of a flow is afforded by the Duffing system*

$$\begin{aligned} \dot{x}(t) &= y(t) \\ \dot{y}(t) &= 0.15\,y(t) - x(t) + x(t)^3 \end{aligned} \tag{2.6}$$

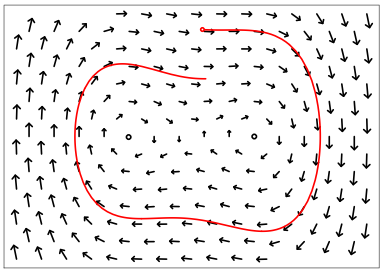
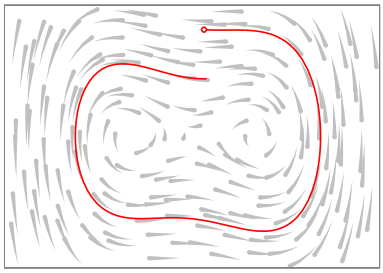
(a)  (b) 

Figure 2.2: (a) The two-dimensional vector field for the Duffing system (2.6), together with a short trajectory segment. (b) The flow lines. Each "comet" represents the same time interval of a trajectory, starting at the tail and ending at the head. The longer the comet, the faster the flow in that region.

plotted in fig. 2.2. The velocity vectors are drawn superimposed over the configuration coordinates $(x(t), y(t))$ *of phase space* $\mathcal{M}$*, but they belong to a different space, the* tangent bundle $\mathbf{T}\mathcal{M}$.

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$$\text{If} \qquad v(x_q) = 0\,, \tag{2.7}$$

x_q is an *equilibrium point* (often referred to as a *stationary*, *fixed*, or *stagnation* point), and the trajectory remains forever stuck at x_q. Otherwise the trajectory passing though x_0 at time $t = 0$ can be obtained by integrating the equations (2.5):

$$x(t) = f^t(x_0) = x_0 + \int_0^t d\tau\, v(x(\tau))\,, \qquad x(0) = x_0\,. \tag{2.8}$$

We shall consider here only the *autonomous* or *stationary* flows, that is flows for which the velocity field v_i is not explicitely dependent on time. A non-autonomous system

$$\frac{dy}{d\tau} = w(y, \tau)\,, \tag{2.9}$$

can always be converted into a system where time does not appear explicitly. To do so, extend the phase space to $(d+1)$-dimensional $x = \{y, \tau\}$, and the vector field to

$$v(x) = \begin{bmatrix} w(y,\tau) \\ 1 \end{bmatrix}\,. \tag{2.10}$$

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The new flow $\dot{x} = v(x)$ is autonomous, and the trajectory $y(\tau)$ can be read off $x(t)$ by ignoring the last component of x.

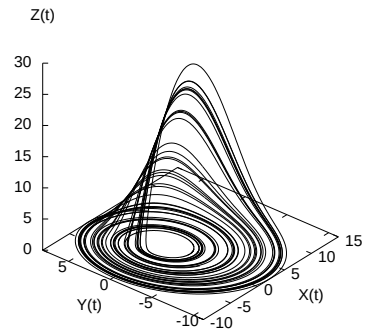

Figure 2.3: A trajectory of the Rössler flow at time $t = 250$. (G. Simon)

Example 2.2 A flow with a strange attractor: *A concrete example of an autonomous flow is the* Rössler system

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c)\,, \qquad a = b = 0.2\,, \quad c = 5.7\,. \end{aligned} \tag{2.11}$$

The system is as simple as they get - it would be linear, were it not for the sole quadratic term zx. Even for so simple a system the nature of long-time solutions is far from obvious. In order to get a feel for what typical solutions look like we need to resort to numerical integration. A typical numerically integrated long-time trajectory is sketched in fig. 2.3. As we shall show in sect. 4.1, for this flow any finite volume of initial conditions shrinks with time, so the flow is contracting. *All trajectories seem to converge to a* strange attractor. *We say "seem", as there exists no proof that such attractor is asymptotically aperiodic - it might well be that what we see is but a long transient on a way to an attractive periodic orbit. For now, accept that fig. 2.3 and similar figures in what follows are examples of "strange attractors".*

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chapter 2.3, p. 40

2.3 Computing trajectories

There is lots of strangeness without beauty
Benny Lautrup

You have not learned dynamics unless you know how to integrate numerically whatever dynamical equations you face. Stated tersely, you need to implement some finite time step prescription for integration of the equations of motion (2.5). The simplest is the Euler integrator which advances the trajectory by $\delta\tau\times$velocity at each time step:

$$x_i \to x_i + \delta\tau v_i(x)\,. \tag{2.12}$$

This might suffice to get you started, but as soon as you need higher numerical accuracy, you will need something better. There are many excellent reference texts and computer programs that can help you learn how to solve differential equations numerically using sophisticated numerical tools, such as pseudo-spectral methods or implicit methods. If a "sophisticated" integration routine takes days and gobbles up terabits of memory, you are using brain-damaged high level software. Try writing a few lines of your own Runge-Kuta code in some mundane everyday language. While you absolutely need to master the requisite numerical methods, this is neither the time nor the place to expand on them; how you learn them is your business. And if you have developed some nice routines for solving problems in this text or can point another student to some, let us know.

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In the next section we dispose of the fear of "infinite-dimensional" dynamical systems - you might prefer to skip the section on first reading.

fast track:
chapter 3, p. 51

2.4 Infinite-dimensional flows

> Romeo: 'Misshapen chaos of well seeming forms!'
> W. Shakespeare, *Romeo and Julliet*, act I, scene I

There is only one honorable cause that would justify sweating through so much formalism - this is but the sharpening of a pencil in order that we may attack the Navier-Stokes equation,

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\nabla p+\eta\left(\nabla^2\mathbf{u}+\frac{1}{3}\nabla(\operatorname{div}\mathbf{u})\right)+\mathbf{f}\,, \tag{2.13}$$

and solve the problem of turbulence. Being realistic, we are not so foolhardy to immediately plunge into *the* problem. Too many dimensions and indices - instead, we start small, in one spatial dimension, $\mathbf{u}\to u$, $\mathbf{u}\cdot\nabla\mathbf{u}\to\frac{1}{2}\partial_x u^2$, assume constant ρ, forget about the pressure p, and so on. This line of reasoning, as well as many other equally sensible threads of thought, such as the amplitude equations obtained via weakly nonlinear stability analysis of steady flows, lead to the essentially same nonlinear PDEs, like the one that we turn to in the next section.

Flows described by partial differential equations are considered infinite dimensional because if one writes them down as a set of ordinary differential

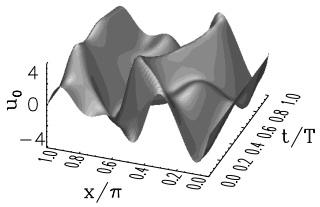

Figure 2.4: Spatiotemporally periodic solution $u_0(x,t)$. We have divided x by π and plotted only the $x > 0$ part, since we work in the subspace of the odd solutions, $u(x,t) = -u(-x,t)$. $N = 16$ Fourier modes truncation with $\nu = 0.029910$. (From ref. [2.7])

equations (ODE) then one needs an infinity of the ordinary kind to represent the dynamics of one equation of the partial kind (PDE). Even though the phase space is infinite dimensional, for many systems of physical interest the global attractor is finite dimensional. We illustrate how this works with a concrete example, the Kuramoto-Sivashinsky system.

in depth:
appendix B, p. 617

2.4.1 Fluttering flame front

The Kuramoto-Sivashinsky system, arising in description of the flame front flutter of gas burning in a cylindrically symmetric burner on your kitchen stove and many other problems of greater import, is one of the simplest partial differential equations that exhibit chaos. It is a dynamical system extended in one spatial dimension, defined by

$$u_t = (u^2)_x - u_{xx} - \nu u_{xxxx}\,. \tag{2.14}$$

In this equation $t \geq 0$ is the time and $x \in [0, 2\pi]$ is the space coordinate. The subscripts x and t denote the partial derivatives with respect to x and t; $u_t = du/dt$, u_{xxxx} stands for 4th spatial derivative of the "height of the flame front" (more correctly the "velocity of the flame front") $u = u(x,t)$ at position x and time t. ν is a "viscosity" parameter; its role is to suppress solutions with fast spatial variations. We take note, as in the Navier-Stokes equation (2.13), of the $u\partial_x u$ "inertial" term, the $\partial_x^2 u$ "diffusive" term (both with a "wrong" sign), etc.

The term $(u^2)_x$ makes this a *nonlinear system*. It is the simplest conceivable nonlinear PDE, playing the role in the theory of spatially extended systems analogous to the role that the x^2 nonlinearity plays in the dynamics of iterated mappings. The time evolution of a solution of the Kuramoto-Sivashinsky system is illustrated by fig. 2.4. How are such solutions computed? The salient feature of such partial differential equations is a theorem

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☞ remark 2.1

saying that for any finite value of the phase-space contraction parameter ν, the asymptotic dynamics is describable by a *finite* set of "inertial manifold" ordinary differential equations.

We are studying a "flame front" $u(x,t) = u(x+2\pi,t)$ periodic on the $x \in [0, 2\pi]$ interval, so a reasonable strategy (but by no means the only one) is to expand it in a discrete spatial Fourier series:

$$u(x,t) = \sum_{k=-\infty}^{+\infty} b_k(t) e^{ikx} \,. \tag{2.15}$$

Since $u(x,t)$ is real, $b_k = b^*_{-k}$. Substituting (2.15) into (2.14) yields the infinite ladder of evolution equations for the Fourier coefficients b_k:

$$\dot{b}_k = (k^2 - \nu k^4) b_k + ik \sum_{m=-\infty}^{\infty} b_m b_{k-m} \,. \tag{2.16}$$

As it follows from this equation that $\dot{b}_0 = 0$, the solution integrated over space is constant in time. We shall consider only the case of this average - the mean value of u - equal to zero, $b_0 = \int dx\, u(x,t) = 0$.

The coefficients b_k are in general complex functions of time t. We can isolate a smaller subspace of the system (2.16) further by considering the case of b_k pure imaginary, $b_k = ia_k$, where a_k are real, with the evolution equations

$$\dot{a}_k = (k^2 - \nu k^4) a_k - k \sum_{m=-\infty}^{\infty} a_m a_{k-m} \,. \tag{2.17}$$

This picks out the subspace of odd solutions $u(x,t) = -u(-x,t)$, so $a_{-k} = -a_k$. By picking this subspace we eliminate the continuous translational symmetry from our consideration; that is probably not an option for an experimentalist, but will do for our purposes.

That is the infinite set of ordinary differential equations promised at the beginning of the section.

The trivial solution $u(x,t) = 0$ is an equilibrium point of (2.14), but that is basically all we know as far as analytical solutions are concerned. You can integrate numerically the Fourier modes (2.17), truncating the ladder of equations to a finite number of modes N, that is, set $a_k = 0$ for $k > N$. In applied mathematics literature such truncation is called a

Galerkin truncation. For the parameter values explored below, $N \leq 16$ truncations were deemed sufficiently accurate. In other words, even though our starting point (2.14) is an infinite-dimensional dynamical system, the asymptotic dynamics unfolds on a finite-dimensional attracting manifold, and so we are back on the familiar territory of sect. 2.2: the theory of a finite number of ODEs applies to this infinite-dimensional PDE as well.

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Once the trajectory is computed in the Fourier space, we can recover and plot the corresponding spatiotemporal pattern $u(x,t)$ over the configuration space using (2.15), as in fig. 2.4.

2.4.2 Fourier modes truncations

Thinking is extra price
Fernando Solla

Consider now the case of initial a_k sufficiently small that the bilinear $a_m a_{k-m}$ terms in (2.17) can be neglected. Then we have a set of decoupled linear equations for a_k whose solutions are exponentials, at most a finite number for which $k^2 > \nu k^4$ growing with time, and the infinity of modes for which $\nu k^4 > k^2$ decaying with time. The growth of the unstable long wavelengths (low $|k|$) excites the short wavelengths through the nonlinear term in (2.17). The excitations thus transferred are dissipated by the strongly damped short wavelengths, and a "chaotic equilibrium" can emerge. The very short wavelengths $|k| \gg 1/\sqrt{\nu}$ will remain small for all times, but the intermediate wavelengths of order $|k| \sim 1/\sqrt{\nu}$ will play an important role in maintaining the dynamical equilibrium. As the damping parameter decreases, the solutions increasingly take on shock front character poorly represented by the Fourier basis, and many higher harmonics may need to be kept in truncations of (2.17).

Hence, while one may truncate the high modes in the expansion (2.17), care has to be exercised to ensure that no modes essential to the dynamics are chopped away. In practice one does this by repeating the same calculation at different truncation cutoffs N, and making sure that inclusion of additional modes has no effect within the accuracy desired. For figures given here, the numerical calculations were performed taking $N = 16$ and the damping parameter value $\nu = 0.029910$, for which the system is chaotic (as far as we can determine that numerically).

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The problem with such high dimensional truncations of the infinite tower of equations (2.17) is that the dynamics is difficult to visualize. The best we can do without much programming is to examine the trajectory's projections onto any three axes a_i, a_j, a_k, as in fig. 2.5.

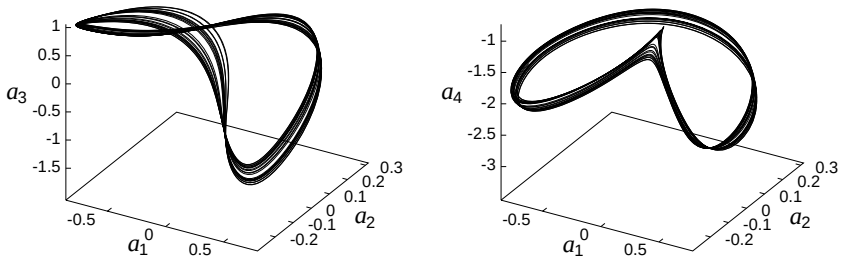

Figure 2.5: Projections of a typical 16-dimensional trajectory onto different 3-dimensional subspaces, coordinates (a) $\{a_1, a_2, a_3\}$, (b) $\{a_1, a_2, a_4\}$. $N = 16$ Fourier modes truncation with $\nu = 0.029910$. (From ref. [2.7].)

Examination of numerical plots such as fig. 2.5 suggests that a more thoughtful approach would be to find a coordinate transformation $y = h(x)$ to a "center manifold", such that in the new, curvilinear coordinates large scale dynamics takes place in (y_1, y_2) coordinates, with exponentially small dynamics in $y_3, y_4 \cdots$. But - thinking is extra price - we do not know how to actually accomplish that.

☞ chapter 6

We can now start to understand the remark on page 38 that for infinite dimensional systems time reversability is not an option: evolution forward in time strongly damps the higher Fourier modes. There is no turning back: if we reverse the time, the infinity of high modes that contract strongly forward in time now explodes, instantly rendering evolution backward in time meaningless. As everything in dynamics, this claim is also wrong, in a subtle way: if the initial $u(x, 0)$ is *in* the non–wandering set (2.2), the trajectory is well defined both forward and backward in time. For practical purposes, this subtlety is not of much use, as any time-reversed numerical trajectory in a finite-mode truncation will explode very quickly, unless special precautions are taken.

Commentary

Remark 2.1 Model ODE and PDE systems. Rössler system was introduced in ref. [2.2], as a simplified set of equations describing time evolution of concentrations of chemical reagents. The Duffing system (2.6) arises in study of electronic circuits. The theorem on finite dimenionality of inertial manifolds of phase-space contracting PDE flows is proven in ref. [2.4]. The Kuramoto-Sivashinsky equation was introduced in refs. [2.5, 2.6]; sect. 2.4 is based on V. Putkaradze's term project (see `wwwcb/extras`), and on the Christiansen *et al.* article [2.7]. How good description of a flame front this equation is need not

concern us here; suffice it to say that such model amplitude equations for interfacial instabilities arise in a variety of contexts - see e.g. ref. [2.8] - and this one is perhaps the simplest physically interesting spatially extended nonlinear system.

Remark 2.2 Diagnosing chaos. In sect. 1.3.1 we have stated that a deterministic system exhibits "chaos" if the dynamics is locally unstable (positive Lyapunov exponent) and globally mixing (positive entropy). In sect. 8.3 we shall define Lyapunov exponents, and discuss their evaluation, but already at this point it would be handy to have a few quick numerical methods to diagnose chaotic dynamics. Laskar's *frequency analysis* method [2.10] is useful for extracting quasi-periodic and weakly chaotic regions of phase space in Hamiltonian dynamics with many degrees of freedom. For references to several other numerical methods, see ref. [2.11].

Résumé

A *dynamical system* – a flow, or an iterated map – is defined by specifying a pair $(\mathcal{M}, f)$, where $\mathcal{M}$ is the phase space and $f : \mathcal{M} \to \mathcal{M}$. The key concepts in exploration of the long time dynamics are the notions of *recurrence* and of the *non–wandering set* of f, the union of all the non-wandering points of $\mathcal{M}$.

Chaotic dynamics with a low dimensional attractor can be visualized as a succession of nearly periodic but unstable motions. In the same spirit, turbulence in spatially extended systems can be described in terms of recurrent spatiotemporal patterns. Pictorially, dynamics drives a given spatially extended system through a repertoire of unstable patterns; as we watch a turbulent system evolve, every so often we catch a glimpse of a familiar pattern. For any finite spatial resolution and finite time the system follows approximately a pattern belonging to a finite repertoire of possible patterns, and the long term dynamics can be thought of as a walk through the space of such patterns. Recasting this image into mathematics is what this book is about.

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