Chapter 4

Local stability

(R. Mainieri and P. Cvitanović)

Topological features of a dynamical system – singularities, periodic orbits, and the ways in which the orbits intertwine – are invariant under a general continuous change of coordinates. More surprisingly, there exist quantities that depend on the notion of metric distance between points, but nevertheless do not change value under a smooth change of coordinates. Local quantities such as stability eigenvalues of equilibria and periodic orbits, and global quantities such as the Lyapunov exponents, metric entropy, and fractal dimensions are examples of such properties of dynamical systems independent of coordinate choice.

We now turn to the first, local class of such invariants, linear stability of flows and maps. This will give us metric information about local dynamics.

4.1 Flows transport neighborhoods

As a swarm of representative points moves along, it carries along and distorts neighborhoods, as sketched in fig. 2.1(b). Deformation of an infinitesimal neighborhood is best understood by considering a trajectory originating near $x_0 = x(0)$ with an initial infinitesimal displacement $\delta x(0)$, and letting the flow transport the displacement $\delta x(t)$ along the trajectory $x(x_0, t) = f^t(x_0)$. The system of linear *equations of variations* for the displacement of the infinitesimally close neighbor $x + \delta x$ follows from the flow equations (2.5) by Taylor expanding to linear order

$$\dot{x}_i + \dot{\delta x}_i = v_i(x + \delta x) \approx v_i(x) + \sum_j \frac{\partial v_i}{\partial x_j} \delta x_j \,.$$

The infinitesimal displacement δx is thus transported along the trajectory $x(x_0, t)$, its time variation given by

$$\frac{d}{dt}\delta x_i(x_0,t) = \sum_j \left.\frac{\partial v_i(x)}{\partial x_j}\right|_{x=x(x_0,t)} \delta x_j(x_0,t)\,. \tag{4.1}$$

As both the displacement and the trajectory always depend on the initial point x_0 and the time t, we shall often abbreviate the notation to $x(x_0,t) \to x(t) \to x$, $\delta x_i(x_0,t) \to \delta x_i(t) \to \delta x$ in what follows. Taken together, the set of equations

$$\dot{x}_i = v_i(x)\,, \quad \dot{\delta x}_i = \sum_j A_{ij}(x)\delta x_j \tag{4.2}$$

governs the dynamics in the tangent bundle $(x, \delta x) \in \mathbf{T}\mathcal{M}$ space obtained by adjoining a d-dimensional fiber, the tangent space $\delta x \in \mathbf{T}_x\mathcal{M}$ to the d-dimensional phase space $x \in \mathcal{M} \subset \mathbb{R}^d$. The matrix of variations

$$A_{ij}(x) = \frac{\partial v_i(x)}{\partial x_j} \tag{4.3}$$

describes the instantaneous rate of shearing of the infinitesimal neighborhood of $x(t)$ by the flow.

Taylor expanding a *finite time* flow to linear order,

$$f_i^t(x_0+\delta x) = f_i^t(x_0) + \sum_j \frac{\partial f_i^t(x_0)}{\partial x_{0j}}\delta x_j + \cdots\,, \tag{4.4}$$

one finds that the linearized neighborhood is transported by the *Jacobian* (or *fundamental*) *matrix*

$$\delta x(t) = \mathbf{J}^t(x_0)\delta x(0)\,, \qquad \mathbf{J}^t_{ij}(x_0) = \left.\frac{\partial x_i(t)}{\partial x_j}\right|_{x=x_0}\,. \tag{4.5}$$

which describes the deformation of an infinitesimal neighborhood at finite time t in the co-moving frame of $x(t)$, that is transformation of the initial, Lagrangian coordinate frame into the current, Eulerian coordinate frame.

As this is a deformation in the linear approximation, you can think of it as a linear deformation of an infinitesimal sphere envelopping x_0 into an

ellipsoid around $x(t)$, described by the eigenvectors and eigenvalues of the Jacobian matrix of the linearized flow. Nearby trajectories separate along the *unstable directions*, approach each other along the *stable directions*, and maintain their distance along the *marginal directions*. In the literature adjectives *neutral* or *indifferent* are often used instead of "marginal".

One of the eigendirections is what one might expect, the direction of the flow itself. To see that, consider two points along a trajectory separated by infinitesimal flight time δt: $\delta x(0) = f^{\delta t}(x_0) - x_0 = v(x_0)\delta t$. At time t later

$$\delta x(t) = f^{t+\delta t}(x_0) - f^t(x_0) = f^{\delta t}(x(t)) - x(t) = v(x(t))\,\delta t\,,$$

while from (4.5)

$$\delta x(t) = \mathbf{J}^t(x_0)\delta x(0)\,.$$

Dividing both equations by δt we observe that $\mathbf{J}^t(x_0)$ transports the velocity vector at x_0 to the velocity vector at $x(t)$ time t later:

$$v(x(t)) = \mathbf{J}^t(x_0)\,v(x_0)\,. \tag{4.6}$$

$v(x(t))$ in general does not point in the same direction as $v(x_0)$, so this is not an eigenvalue condition for $\mathbf{J}^t$; Jacobian matrix computed for an arbitrary segment of an arbitrary trajectory has no invariant meaning. As the eigenvalues of $\mathbf{J}^t$ have invariant meaning only for periodic orbits, we postpone their interpretation to sect. 4.5. However, already at this stage we see that if the orbit is periodic, $x(T) = x(0)$, $v(x_0)$ is an eigenvector of the Jacobian matrix $\mathbf{J}^T(x_0)$ with a unit eigenvalue.

As we started by assuming that we know the equations of motion, from (4.3) we also know $\mathbf{A}$, the instantaneous rate of shear of an infinitesimal neighborhood $\delta x_i(t)$ of the trajectory $x(t)$. What we do not know is the finite time deformation matrix $\mathbf{J}^T$. Our next task is to relate the two.

We are interested in smooth, differentiable flows. If a flow is smooth, in a sufficiently small neighborhood it is essentially linear. Hence the next section, which might seem an embarrassment (what is a section on *linear* flows doing in a book on *non*linear dynamics?), offers a firm stepping stone on the way to understanding nonlinear flows. If you know your eigenvalues and eigenvectors, you may prefer to fast forward here.

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4.2 Linear flows

Linear fields are the simplest of vector fields. They lead to linear differential equations which can be solved explicitly, with solutions which are good for all times. The phase space for linear differential equations is $\mathcal{M} = \mathbb{R}^d$, and the equations of motion (2.5) are written in terms of a vector x and a constant matrix of variations $\mathbf{A}$ as

$$\dot{x} = v(x) = \mathbf{A}x\,. \tag{4.7}$$

Solving this equation means finding the phase space trajectory

$$x(t) = (x_1(t), x_2(t), \ldots, x_d(t))$$

passing through the point x_0.

If $x(t)$ is a solution with $x(0) = x_0$ and $x(t)'$ another solution with $x(0)' = x_0{}'$, then the linear combination $ax(t) + bx(t)'$ with $a, b \in \mathbb{R}$ is also a solution, but now starting at the point $ax_0 + bx_0{}'$. At any instant in time, the space of solutions is a d-dimensional vector space, which means that one can find a basis of d linearly independent solutions. How do we solve the linear differential equation (4.7)? If instead of a matrix equation we have a scalar one, $\dot{x} = ax$, with a a real number, then the solution is

$$x(t) = e^{ta}x(0)\,. \tag{4.8}$$

In order to solve the matrix case, it is helpful to re-derive the solution (4.8) by studying what happens for a short time step Δt. If at time 0 the position is $x(0)$, then

$$\frac{x(0+\Delta t) - x(0)}{\Delta t} = ax(0)\,, \tag{4.9}$$

which we iterate m times to obtain

$$x(t) \approx \left(1 + \frac{t}{m}a\right)^m x(0)\,. \tag{4.10}$$

The term in the parenthesis acts on the initial condition $x(0)$ and evolves it to $x(t)$ by taking m small time steps $\Delta t = t/m$. As $m \to \infty$, the term

in the parenthesis converges to e^{ta}. Consider now the matrix version of equation (4.9):

$$\frac{x(\Delta t) - x(0)}{\Delta t} = \mathbf{A}x(0)\,. \tag{4.11}$$

Representative point x is now a vector in $\mathbb{R}^d$ acted on by the matrix $\mathbf{A}$, as in (4.7). Denoting by $\mathbf{1}$ the identity matrix, and repeating the steps (4.9) and (4.10) we obtain the Euler formula for the exponential of a matrix

$$x(t) = \lim_{m\to\infty} \left(\mathbf{1} + \frac{t}{m}\mathbf{A}\right)^m x(0) = e^{t\mathbf{A}}x(0)\,. \tag{4.12}$$

We will use this expression as the *definition* of the exponential of a matrix.

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4.2.1 Stability eigenvalues, diagonal case

How do we compute the exponential (4.12)? Should we be so lucky that $\mathbf{A}$ happens to be a diagonal matrix $\mathbf{A}_D$ with eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_d)$, the exponential is simply

$$\mathbf{J}^t = e^{t\mathbf{A}_D} = \begin{pmatrix} e^{t\lambda_1} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & e^{t\lambda_d} \end{pmatrix}\,. \tag{4.13}$$

Suppose next that $\mathbf{A}$ is diagonalizable and that $\mathbf{U}$ is the matrix that brings it to its diagonal form $\mathbf{A}_D = \mathbf{U}\mathbf{A}\mathbf{U}^{-1}$. The transformation $\mathbf{U}$ is a linear coordinate transformation which rotates, skews, and possibly flips the coordinate axis of the vector space. Then $\mathbf{J}$ can also be brought to a diagonal form:

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$$\mathbf{J}^t = e^{t\mathbf{A}} = \mathbf{U}^{-1}e^{t\mathbf{A}_D}\mathbf{U}\,. \tag{4.14}$$

In either case, the action of both $\mathbf{A}$ and $\mathbf{J}$ is very simple; the axes of orthonormal coordinate system where $\mathbf{A}$ is diagonal are also the eigendirections of both $\mathbf{A}$ and $\mathbf{J}^t$, and under the flow the neighborhood is deformed by a multiplication by an eigenvalue factor for each coordinate axis.

4.2.2 Complex stability eigenvalues

As $\mathbf{A}$ has only real entries, it will in general have either real eigenvalues, or complex conjugate pairs of eigenvalues. That is not surprising, but also the corresponding eigenvectors can be either real or complex. All coordinates used in defining the flow are real numbers, so what is the meaning of a *complex* eigenvector?

To develop some intuition about that, let us work out the behavior for the simplest nontrivial case, the case where $\mathbf{A}$ is a [2×2] matrix

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} . \tag{4.15}$$

The eigenvalues λ_1, λ_2 of $\mathbf{A}$ are the roots

$$\lambda_{1,2} = \frac{1}{2}\left(\mathrm{tr}\,\mathbf{A} \pm \sqrt{(\mathrm{tr}\,\mathbf{A})^2 - 4\det\mathbf{A}}\right) \tag{4.16}$$

of the characteristic equation

$$\begin{aligned} \det(\mathbf{A} - z\mathbf{1}) &= (\lambda_1 - z)(\lambda_2 - z) = 0\,, \\ \left|\left| \begin{matrix} A_{11} - z & A_{12} \\ A_{21} & A_{22} - z \end{matrix} \right|\right| &= z^2 - (A_{11} + A_{22})\, z + (A_{11}A_{22} - A_{12}A_{21})\,. \end{aligned} \tag{4.17}$$

The qualitative behavior of the exponential of $\mathbf{A}$ for real eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ will differ from the case that they form a complex conjugate pair,

$$\lambda_1 = \lambda + i\theta\,, \quad \lambda_2 = \lambda_1^* = \lambda - i\theta\,.$$

These two possibilities are refined further into sub-cases depending on the signs of the real part. The matrix might have only one eigenvector, or two linearly independent eigenvectors, which may or may not be orthogonal. Along each of these directions the motion is of the form $\exp(t\lambda_k)x_k$. If the exponent λ_k is positive, then the component x_k will grow; if the exponent λ_k is negative, it will shrink.

We sketch the full set of possibilities in fig. 4.1(a), and work out in detail the case when $\mathbf{A}$ can be brought to the diagonal form. Then the solution (4.12) to the differential equation (4.7) can be written either as

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} , \tag{4.18}$$

or

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{t\lambda} \begin{pmatrix} e^{it\theta} & 0 \\ 0 & e^{-it\theta} \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} . \tag{4.19}$$

In the case $\lambda_1 > 0$, $\lambda_2 < 0$, x_1 grows exponentially with time, and x_2 contracts exponentially. This behavior, called a *saddle*, is sketched in fig. 4.1(b), as are the remaining possibilities: in/out nodes, inward/outward spirals, and the center. Spirals arise from taking a real part of the action of $\mathbf{J}^t$ on a complex eigenvector. The magnitude of $|x(t)|$ diverges exponentially the $\lambda > 0$, and contracts toward 0 if the $\lambda < 0$, while the imaginary phase θ controls its oscillations.

4.2.3 General, nonsymmetric Jacobian matrix

In general $\mathbf{J}^t$ is neither diagonal, nor diagonalizable, nor constant along the trajectory. Still, any matrix, including $\mathbf{J}^t$, can be expressed in the singular value decomposition form

$$\mathbf{J} = \mathbf{U}\mathbf{D}\mathbf{V}^T$$

where $\mathbf{D}$ is diagonal, and $\mathbf{U}$, $\mathbf{V}$ are orthogonal matrices. The diagonal elements Λ_1, Λ_2, ..., Λ_d of $\mathbf{D}$ are called the *stability eigenvalues*.

Under the action of the flow an infinitesimally small ball of initial points is deformed into an ellipsoid: Λ_i is the relative stretching of the ith principal axis of the ellipsoid, the columns of the matrix $\mathbf{V}$ are the principal axes $\mathbf{e}_i$ of stretching in the Lagrangian coordinate frame, and the orthogonal matrix $\mathbf{U}$ gives the orientation of the ellipse in the Eulerian coordinates.

Now that we have some feeling for the qualitative behavior of eigenvectors and eigenvalues, we are ready to return to the general case: nonlinear flows.

4.3 Stability of flows

How do you determine the eigenvalues of the finite time local deformation $\mathbf{J}^t$ for a general nonlinear smooth flow? The Jacobian matrix is computed by integrating the equations of variations (4.2)

$$x(t) = f^t(x_0) , \quad \delta x(x_0, t) = \mathbf{J}^t(x_0)\delta x(x_0, 0) . \tag{4.20}$$

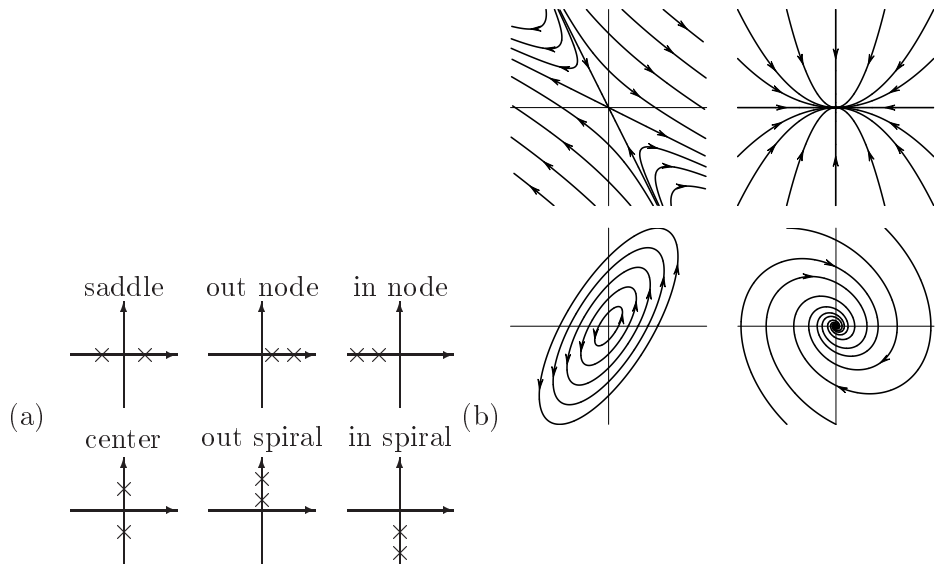

Figure 4.1: (a) Qualitatively distinct types of eigenvalues of a [2×2] stability matrix. (b) Streamlines for several typical 2-dimensional flows.

The equations of variations are linear, so we should be able to integrate them - but in order to make sense of the answer, we derive it step by step.

4.3.1 Stability of equilibria

For a start, consider the case where x_q is an equilibrium point (2.7). Expanding around the equilibrium point x_q, using the fact that the matrix $\mathbf{A} = \mathbf{A}(x_q)$ in (4.2) is constant, and integrating,

$$f^t(x) = x_q + e^{\mathbf{A}t}(x - x_q) + \cdots , \tag{4.21}$$

we verify that the simple formula (4.12) applies also to the Jacobian matrix of an equilibrium point, $\mathbf{J}^t(x_q) = e^{\mathbf{A}t}$.

Example 4.1 Why does a flame front flutter? *The Kuramoto-Sivashinsky flat frame-front $u(x,t) = 0$ is an equilibrium point of (2.14). The matrix of variations (4.3) follows from (2.17)*

$$A_{kj}(a) = \frac{\partial v_k(x)}{\partial a_j} = (k^2 - \nu k^4)\delta_{kj} - 2k a_{k-j} \, . \tag{4.22}$$

For the $u(x,t) = 0$ equilibrium solution the matrix of variations is diagonal, and as in (4.13), so is the Jacobian matrix $\mathbf{J}^t(0)_{kj} = \delta_{kj} e^{(k^2 - \nu k^4)t}$.

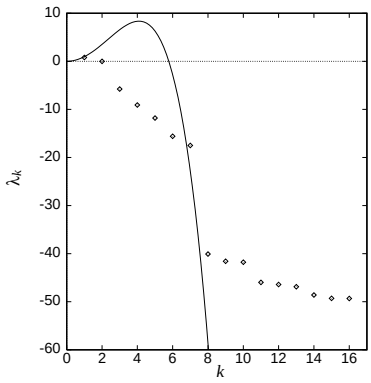

Figure 4.2: Lyapunov exponents $\lambda_{\overline{1},k}$ versus k for the least unstable spatio-temporally periodic orbit $\overline{1}$ of the Kuramoto-Sivashinsky system, compared with the stability exponents of the $u(x,t) = 0$ stationary solution, $\lambda_k = k^2 - \nu k^4$. $\lambda_{\overline{1},k}$ for $k \geq 8$ fall below the numerical accuracy of integration and are not meaningful. $N = 16$ Fourier modes, $\nu = 0.029924$, chaotic regime. The cycle $\overline{1}$ was computed using methods of chapter 14. (From ref. [2.7])

For $\nu > 1$, $u(x,t) = 0$ is the globally attractive stable equilibrium. With the "viscosity" $\nu = 1$ and below, the dynamics goes through a rich sequence of bifurcations on which we shall not dwell here. The $|k| < 1/\sqrt{\nu}$ long wavelength perturbations of the flat-front equilibrium are linearly unstable, while all $|k| > 1/\sqrt{\nu}$ short wavelength perturbations are strongly contractive. The high k eigenvalues, corresponding to rapid variations of the flame front, decay so fast that the corresponding eigendirections have no physical meaning. To illustrate the rapid contraction in the non-leading eigendirections we plot in fig. 4.2 the eigenvalues of the equilibrium in the unstable regime, for relatively low "viscosity" ν, and compare them with the stability eigenvalues of the least unstable cycle for the same value of ν. The equilibrium solution is very unstable, in 5 eigendirections, the least unstable cycle only in one, but for $k > 7$ the rate of contraction is so strong that higher eigendirections are numerically meaningless for either solution.

4.3.2 Stability of trajectories

Next, consider the case of a general, non-stationary trajectory $x(t)$. The exponential of a constant matrix can be defined either by its Taylor series expansion, or in terms of the Euler limit (4.12):

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$$e^{t\mathbf{A}} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k \tag{4.23}$$

$$= \lim_{m\to\infty} \left(\mathbf{1} + \frac{t}{m}\mathbf{A}\right)^m . \tag{4.24}$$

Taylor expansion is fine if $\mathbf{A}$ is a constant matrix. However, only the second, tax-accountant's discrete step definition of exponential is appropriate for the task at hand, as for a dynamical system the local rate of neighborhood distortion $\mathbf{A}(x)$ depends on where we are along the trajectory. The linearized neighborhood is multiplicatively deformed along the flow, and the m discrete time steps approximation to $\mathbf{J}^t$ is therefore given by

generalization of the Euler product (4.24) to

$$\mathbf{J}^t = \lim_{m\to\infty} \prod_{n=m}^{1} (\mathbf{1} + \Delta t \mathbf{A}(x_n)) = \lim_{m\to\infty} \prod_{n=m}^{1} e^{\Delta t\, \mathbf{A}(x_n)} \tag{4.25}$$

where $\Delta t = (t - t_0)/m$, $x_n = x(t_0 + n\Delta t)$. The two formulas for $\mathbf{J}^t$ agree to the leading order in Δt, and the $m \to \infty$ limit of this procedure is the integral

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$$\mathbf{J}^t_{ij}(x_0) = \left[\mathbf{T} e^{\int_0^t d\tau \mathbf{A}(x(\tau))}\right]_{ij} . \tag{4.26}$$

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where $\mathbf{T}$ stands for time-ordered integration. This formula for $\mathbf{J}$ is the main result of this chapter.

It is evident from the time-ordered product structure (4.25) that the Jacobian matrices are multiplicative along the flow,

$$\mathbf{J}^{t+t'}(x) = \mathbf{J}^{t'}(x')\mathbf{J}^t(x), \qquad \text{where } x' = f^t(x)\,. \tag{4.27}$$

The formula (4.25) is a matrix generalization of the crude Euler integrator (2.12), neither smart not accurate. Much better numerical accuracy is obtained by the following observation. To linear order in Δt, $(\mathbf{J}^{t+\Delta t} - \mathbf{J}^t)$ equals $\Delta t\, \mathbf{A}\mathbf{J}^t$, so the Jacobian matrix itself satisfies the linearized equation (4.1)

$$\frac{d}{dt}\mathbf{J}^t(x) = \mathbf{A}(x)\,\mathbf{J}^t(x)\,, \qquad \text{with the initial condition } \mathbf{J}^0(x) = \mathbf{1}\,. \tag{4.28}$$

Given a numerical routine for integrating the equations of motion, evaluation of the Jacobian matrix requires minimal additional programming effort; one simply extends the d-dimensional integration routine and integrates concurrently with $f^t(x)$ the d^2 elements of $\mathbf{J}^t(x)$.

"Simply" is perhaps too glib. Integration will work for short finite times, but for exponentially unstable flows one quickly runs into numerical over/underflow problems, and further thought will be required.

4.4 Stability of maps

The transformation of an infinitesimal neighborhood of a trajectory under map iteration follows from Taylor expanding the iterated mapping at

discrete time n to linear order, as in (4.4). The linearized neighborhood is transported by the Jacobian matrix evaluated at a discrete set of times $n = 1, 2, \ldots,$

$$\mathbf{J}^n_{ij}(x_0) = \left.\frac{\partial f^n_i(x)}{\partial x_j}\right|_{x=x_0} . \tag{4.29}$$

This matrix is in the literature sometimes called the *fundamental* matrix. As the simplest example, consider a 1-dimensional map. The chain rule yields stability of the nth iterate

$$\Lambda_n = \frac{d}{dx} f^n(x) = \prod_{m=0}^{n-1} f'(x_m) , \quad x_m = f^m(x_0) . \tag{4.30}$$

The 1-step product formula for the stability of the nth iterate of a d-dimensional map

$$\mathbf{J}^n(x_0) = \prod_{m=n-1}^{0} \mathbf{J}(x_m) , \quad \mathbf{J}(x)_{kl} = \frac{\partial}{\partial x_l} f_k(x) , \quad x_{m,j} = f^m(x_0)_j \tag{4.31}$$

follows from the chain rule for matrix derivatives

$$\frac{\partial}{\partial x_i} f_j(f(x)) = \sum_{k=1}^{d} \left.\frac{\partial}{\partial y_k} f_j(y)\right|_{y=f(x)} \frac{\partial}{\partial x_i} f_k(x) .$$

The $[d\times d]$ Jacobian matrix $\mathbf{J}^n$ for a map is evaluated by multiplication along the n points $x_0, x_1, x_2, \ldots x_{n-1}$ on the trajectory of x_0, with $\mathbf{J}(x)$ the single time step Jacobian matrix.

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Example 4.2 Hénon map Jacobian matrix. *For the Hénon map (3.10) the Jacobian matrix for nth iterate of the map is*

$$\mathbf{J}^n(x_0) = \prod_{m=n}^{1} \begin{pmatrix} -2ax_m & b \\ 1 & 0 \end{pmatrix} , \qquad x_m = f_1^m(x_0, y_0) . \tag{4.32}$$

The determinant of the Hénon one time step Jacobian matrix (4.32) is constant,

$$\det \mathbf{J} = \Lambda_1 \Lambda_2 = -b \tag{4.33}$$

so in this case only one eigenvalue $\Lambda_1 = -b/\Lambda_2$ needs to be determined. This is not an accident; a constant jacobian was one of desiderata that led Hénon to constructing the map of this particular form.

4.5 Stability of periodic orbits

As noted on page 36, a trajectory can be stationary, periodic or aperiodic. For chaotic systems almost all trajectories are aperiodic – nevertheless, the stationary and the periodic orbits will turn out to be the key to unraveling chaotic dynamics. Here we note a few of the properties that makes them so precious to a theorist.

An obvious virtue of periodic orbits is that they are *topological* invariants: a fixed point is a fixed point in any coordinate choice, and similarly a periodic orbit is a periodic orbit in any representation of the dynamics. Any re-parametrization of a dynamical system that preserves its topology has to preserve topological relations between periodic orbits, such as their relative inter-windings and knots. So mere existence of periodic orbits suffices to partially organize the spatial layout of a non–wandering set. No less important, as we shall now show, is the fact that cycle stability eigenvalues are *metric* invariants: they determine the relative sizes of neighborhoods in a non–wandering set.

To prove this, we start by noting that due to the multiplicative structure (4.27) of Jacobian matrices, the stability of the rth repeat of a prime cycle p of period T_p is

$$\mathbf{J}^{rT_p}(x_0) = \mathbf{J}^{T_p}(f^{rT_p}(x_0)) \cdots \mathbf{J}^{T_p}(f^{T_p}(x_0))\mathbf{J}^{T_p}(x_0) = \mathbf{J}_p(x_0)^r \,, \quad (4.34)$$

where $\mathbf{J}_p(x_0) = \mathbf{J}^{T_p}(x_0)$ is the stability matrix for a single traversal of the prime cycle p, $x_0 \in p$ is any point on the cycle, and $f^{rT_p}(x_0) = x_0$ as $f^t(x_0)$ returns to x_0 every multiple of the period T_p. Hence, it suffices to restrict our considerations to the stability of prime cycles.

4.5.1 Stability eigenvalues, stability exponents

We sort the *stability eigenvalues* $\Lambda_{p,1}$, $\Lambda_{p,2}$, ..., $\Lambda_{p,d}$ of the $[d\times d]$ Jacobian matrix $\mathbf{J}_p$ evaluated on the p cycle into sets $\{e, m, c\}$

$$\begin{aligned} \text{expanding:} \quad & \{\Lambda_p\}_e &&= \{\Lambda_{p,i} : |\Lambda_{p,i}| > 1\} \\ \text{marginal:} \quad & \{\Lambda_p\}_m &&= \{\Lambda_{p,i} : |\Lambda_{p,i}| = 1\} \\ \text{contracting:} \quad & \{\Lambda_p\}_c &&= \{\Lambda_{p,i} : |\Lambda_{p,i}| < 1\} \,. \end{aligned} \quad (4.35)$$

and denote by Λ_p (no spatial index) the product of expanding eigenvalues

$$\Lambda_p = \prod_e \Lambda_{p,e} \,. \quad (4.36)$$

As $\mathbf{J}_p$ is a real matrix, complex Λ_i always come in $\Lambda_{p,i}$, $\Lambda_{p,i+1} = \Lambda^*_{p,i}$ pairs, and Λ_p is always real.

Cycle *stability exponents* are the stretching/contraction rates per unit time

$$\lambda_{p,i} = \frac{1}{T_p} \ln |\Lambda_{p,i}| \ . \tag{4.37}$$

This definition is motivated by the form of the stability exponents for the linear case, for example (4.13), as well as the fact that exponents so defined can be interpreted as the Lyapunov exponents evaluated on the prime cycle p. As in the three cases of (4.35), we sort the stability exponents into three sets

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$$\begin{aligned} \text{expanding:} \quad & \{\lambda_p\}_e &&= \{\lambda_{p,i} : \ \lambda_{p,i} > 0\} \\ \text{elliptic:} \quad & \{\lambda_p\}_m &&= \{\lambda_{p,i} : \ \lambda_{p,i} = 0\} \\ \text{contracting:} \quad & \{\lambda_p\}_c &&= \{\lambda_{p,i} : \lambda_{p,i} < 0\} \, . \end{aligned} \tag{4.38}$$

A periodic orbit p of a d-dimensional flow or a map is *stable* if all its stability exponents (other than the vanishing longitudinal stability exponent, to be explained in sect. 4.5.3 below) are strictly negative, $|\lambda_{p,i}| < 0$. The region of system parameter values for which a periodic orbit p is stable is called the *stability window* of p. The subset $\mathcal{M}$ of initial points that are asymptotically attracted to p (for a fixed set of system parameter velues) is called the *basin of attraction* of p.

If *all* stability exponents (other than the vanishing longitudinal stability exponent) of *all* periodic orbits of a flow are strictly bounded away from zero, $|\lambda_i| \geq \lambda_{min} > 0$, the flow is said to be *hyperbolic*. Otherwise the flow is said to be *nonhyperbolic*.

As we often do care about the sign of $\Lambda_{p,i}$ and, if $\Lambda_{p,i}$ is complex, its phase

$$\Lambda_{p,j} = \pm e^{T_p(\lambda_{p,j} \pm i\theta_{p,j})} \, , \tag{4.39}$$

☞ sect. 5.1.1

and keeping track of those by case-by-case enumeration is a self-inflicted, unnecessary nuisance (followed by much of the literature), almost all of our formulas will be stated in terms of the stability eigenvalues Λ rather than in the terms of the overall signs, stability exponents λ_i and phases θ_i.

Example 4.3 1-dimensional maps. *The simplest example of cycle stability is afforded by 1-dimensional maps. The stability of a prime cycle p follows from the chain rule (4.30) for stability of the n_pth iterate of the map*

$$\Lambda_p = \frac{d}{dx_0} f^{n_p}(x_0) = \prod_{m=0}^{n_p-1} f'(x_m)\,, \quad x_m = f^m(x_0)\,, \tag{4.40}$$

where the initial $x_0 \in p$ can be any of the periodic points in the p cycle.

A critical point *x_c is a value of x for which the mapping $f(x)$ has vanishing derivative, $f'(x_c) = 0$. For future reference we note that a periodic orbit of a 1-dimensional map is* stable *if*

$$|\Lambda_p| = \left|f'(x_{n_p})f'(x_{n_p-1})\cdots f'(x_2)f'(x_1)\right| < 1\,,$$

and superstable *if the orbit includes a critical point, so that the above product vanishes. For a stable periodic orbit of period n the slope of the nth iterate $f^n(x)$ evaluated on a periodic point x (fixed point of the nth iterate) lies between -1 and 1. If $|\Lambda_p| > 1$, cycle p is* unstable.

4.5.2 Cycle stabilities are cycle invariants

The 1-dimensional map cycle stability Λ_p is a product of derivatives over all cycle points around the cycle, and is therefore independent of which periodic point is chosen as the initial one. In higher dimensions the form of the Jacobian matrix $\mathbf{J}_p(x_0)$ in (4.34) does depend on the choice of coordinates and the initial point $x_0 \in p$. Nevertheless, as we shall now show, the cycle stability *eigenvalues* are intrinsic property of a cycle also for multi-dimensional flows. Consider the ith eigenvalue, eigenvector pair $(\Lambda_{p,i}, \mathbf{e}_i)$ computed from $\mathbf{J}_p$ evaluated at a cycle point,

$$\mathbf{J}_p(x)\mathbf{e}_i(x) = \Lambda_{p,i}\mathbf{e}_i(x)\,, \quad x \in p\,. \tag{4.41}$$

Consider another point on the cycle a time t later, $x' = f^t(x)$ whose Jacobian matrix is $\mathbf{J}_p(x')$. By the group property (4.27), $\mathbf{J}^{T_p+t} = \mathbf{J}^{t+T_p}$, and the Jacobian matrix at x' can be written either as

$$\mathbf{J}^{T_p+t}(x) = \mathbf{J}^{T_p}(x')\mathbf{J}^t(x) = \mathbf{J}_p(x')\mathbf{J}^t(x)\,, \quad \text{or} \quad \mathbf{J}^{t+T_p}(x) = \mathbf{J}^t(x)\mathbf{J}_p(x)\,.$$

Multiplying (4.41) by $\mathbf{J}^t(x)$, we find that the Jacobian matrix evaluated at x' has the same eigenvalue,

$$\mathbf{J}_p(x')\mathbf{e}_i(x') = \Lambda_{p,i}\mathbf{e}_i(x')\,, \quad \mathbf{e}_i(x') = \mathbf{J}^t(x)\mathbf{e}_i(x)\,, \tag{4.42}$$

but with the eigenvector $\mathbf{e}_i$ transported along the flow $x \to x'$ to $\mathbf{e}_i(x') = \mathbf{J}^t(x)\mathbf{e}_i(x)$. Hence, $\mathbf{J}_p$ evaluated anywhere along the cycle has the same set of stability eigenvalues $\{\Lambda_{p,1}, \Lambda_{p,2}, \cdots \Lambda_{p,d}\}$. As quantities such as $\mathrm{tr}\, \mathbf{J}_p(x)$, $\det \mathbf{J}_p(x)$ depend only on the eigenvalues of $\mathbf{J}_p(x)$ and not on the starting point x, in expressions such as $\det\left(\mathbf{1} - \mathbf{J}_p^r(x)\right)$ we may omit reference to any particular cycle point x:

$$\det\left(\mathbf{1} - \mathbf{J}_p^r(x)\right) = \det\left(\mathbf{1} - \mathbf{J}_p^r\right) . \tag{4.43}$$

We postpone the proof that the cycle stability eigenvalues are smooth conjugacy invariants of the flow to sect. 6.4.

4.5.3 Marginal eigenvalues

The presence of marginal eigenvalues signals either an invariance of the flow (which you should immediately exploit to simplify the problem), or a non-hyperbolicity of a flow (source of much pain, hard to avoid).

☞ chapter 18

A periodic orbit of a continuous flow always has at least one marginal eigenvalue. As $\mathbf{J}^t(x)$ transports the velocity field $v(x)$ by (4.6), after a complete period

$$\mathbf{J}_p(x)v(x) = v(x)\,, \tag{4.44}$$

so a periodic orbit of a *flow* always has an eigenvector $\mathbf{e}_{||}(x) = v(x)$ parallel to the local velocity field with the unit eigenvalue

$$\Lambda_{p,||} = 1\,. \tag{4.45}$$

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The continuous invariance that gives rise to this marginal eigenvalues is the invariance of a cycle under a translation of its points along the cycle. As we shall see in sect. 11.3, this marginal stability direction can be eliminated by "fixing the gauge", that is by cutting the cycle by a Poincaré section and eliminating the continuous flow Jacobian matrix in favor of the Jacobian matrix of the Poincaré section return map.

4.6 Neighborhood of a cycle

☞ sect. 8.3
☞ remark 8.3

The Jacobian of the flow (or the *sum* of stability exponents) is easily evaluated.

Consider $\det \mathbf{J}^t(x_0) = \prod_{i=1}^d \Lambda_i(x_0,t)$, the product of the stability eigenvalues. We shall refer to this determinant as the *Jacobian* of the flow. By means of the time-ordered product (4.25) and the identity $\ln\det M = \mathrm{tr}\,\ln M$ the Jacobian is given by

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$$\det \mathbf{J}^t(x_0) = e^{\int_0^t d\tau\, \mathrm{tr}\, \mathbf{A}(x(\tau))} = e^{\int_0^t d\tau\, \partial_i v_i(x(\tau))}\,. \tag{4.46}$$

As the divergence $\partial_i v_i$ is a scalar quantity, the integral in the exponent needs no time ordering. All we need to do is to evaluate the time average

$$\langle \partial_i v_i \rangle_t = \frac{1}{t} \ln \left| \prod_{i=1}^d \Lambda_i(x_0,t) \right| = \sum_{i=1}^d \lambda_i(x_0,t) = \frac{1}{t}\int_0^t d\tau \sum_{i=1}^d A_{ii}(x(\tau)) \tag{4.47}$$

along the trajectory. If the flow is not singular (for example, the trajectory does not run head-on into the Coulomb $1/r$ singularity), the matrix of variations elements $|A_{ij}|$ are everywhere bounded in magnitude, and so is the trace $\sum_i A_{ii}$. The time integral in (4.47) grows at most linearly with t, hence $\langle \partial_i v_i \rangle_t$ is bounded, regardless of how large the time t might be, and numerical estimates of the $t\to\infty$ limit $\langle \partial_i v_i \rangle$ are not marred by any exponential blowups.

Even if we were to insist on extracting $\langle \partial_i v_i \rangle$ from (4.25) by first multiplying stability matrices along the flow, and then taking the logarithm, we can avoid exponential blowups in $\mathbf{J}^t$ by using the multiplicative structure (4.27), $\det \mathbf{J}^{t'+t}(x_0) = \det \mathbf{J}^{t'}(x')\det \mathbf{J}^t(x_0)$ to restart with $\mathbf{J}^0(x') = \mathbf{1}$ whenever the eigenvalues of $\mathbf{J}^t(x_0)$ start getting out of hand. In numerical evaluations of Lyapunov exponents, $\lambda_i = \lim_{t\to\infty}\lambda_i(x_0,t)$, sum rule (4.47) can serve as a helpful check on the accuracy of the computation.

sect. 8.3

$\langle \partial_i v_i \rangle$ is also an important physical characterization of the flow - it tells us what is the behavior of a phase space volume in the infinitesimal neighborhood of the trajectory. If $\partial_i v_i < 0$, the flow is *locally contracting*, and the trajectory might be falling into an attractor. If $\partial_i v_i = 0$, the flow preserves phase space volume and $\det \mathbf{J}^t = \mathbf{1}$. A flow with this property is called *incompressible*. An important class of such flows are the Hamiltonian flows to which we turn in sect. 5.1.1.

But before we can get to that, the alert student, pipes up. He does not like our definition of the Jacobian matrix in terms of the time-ordered exponential (4.26). Depending on the signs of stability eigenvalues, the left hand side of (4.46) can be either positive or negative. But the right hand side is an exponential of a real number, and that can only be positive. What gives? As we shall see much later on in this text, in discussion of

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topological topological indices arising in semiclassical quantization, this is not at all a dumb question.

in depth:
appendix K.1, p. 697

in depth:
appendix C.1, p. 621

4.6.1 There goes the neighborhood

In what follows, our task will be to determine the size of a *neighborhood* of $x(t)$, and that is why we care about the stability eigenvalues, and especially the unstable (expanding) ones. Nearby points aligned along the stable (contracting) directions remain in the neighborhood of the trajectory $x(t) = f^t(x_0)$; the ones to keep an eye on are the points which leave the neighborhood along the unstable directions. The sub-volume $|\mathcal{M}_i| = \prod_i^e \Delta x_i$ of the set of points which get no further away from $f^t(x_0)$ than L, the typical size of the system, is fixed by the condition that $\Delta x_i \Lambda_i = O(L)$ in each expanding direction i. Hence the neighborhood size scales as $\propto 1/|\Lambda_p|$ where Λ_p is the product of expanding eigenvalues (4.36) only; contracting ones play a secondary role. So secondary that even infinity of them (for example, the infinity of contracting eigendirections of the spatiotemporal dynamics of sect. 2.4.1) will not matter.

So the physically important information is carried by the expanding sub-volume, not the total volume computed so easily in (4.47). That is also the reason why the dissipative and the Hamiltonian chaotic flows are much more alike than one would have naively expected for "compressible" *vs.* "incompressible" flows. What matters are the expanding directions. Whether the contracting eigenvalues are inverses of the expanding ones or not is of secondary importance. As long as the number of unstable directions is finite, the same theory will describe finite-dimensional ODEs and infinite-dimensional PDEs.

Résumé

A neighborhood of a trajectory deforms as it is transported by the flow. In the linear approximation, the matrix of variations $\mathbf{A}$ describes this shearing of an infinitesimal neighborhood in an infinitesimal time step. The shearing after a finite time t is described by the Jacobian matrix

$$\mathbf{J}^t(x_0) = \mathbf{T} e^{\int_0^t d\tau \mathbf{A}(x(\tau))} ,$$

where $\mathbf{T}$ stands for the *time-ordered* integration. Its eigenvalues and eigendirections describe deformation of an initial infinitesimal sphere of neighboring trajectories into an ellipsoid finite time t later. Nearby trajectories separate exponentially along unstable directions, approach each other along stable directions, and change slowly (algebraically) their distance along marginal directions. The Jacobian matrix $\mathbf{J}^t$ is in general neither symmetric, nor diagonalizable by a rotation, nor do its (left or right) eigenvectors define an orthonormal coordinate frame. Furthermore, while the stability matrices are multiplicative along the flow, in dimensions higher than one their eigenvalues in general are not. This lack of multiplicativity has important repercussion both for classical dynamics and quantum dynamics.

☞ appendix H.1

Periodic orbits play a central role in any invariant characterization of the dynamics, because (a) their existence and inter-relations are *topological*, coordinate choice independent property of the dynamics, and (b) their stability eigenvalues are metric invariants. We shall show in chapter 9 that extending their local stability eigendirections into stable and unstable manifolds yields also important global information about the topological organization of the phase space.

References

[4.1] J.-L. Thiffeault, "Derivatives and constraints in chaotic flows: Asymptotic behaviour and a numerical method", preprint (2001).