Appendix I

Convergence of spectral determinants

I.1 Curvature expansions: geometric picture

If you has some experience with numerical estimates of fractal dimensions, you will note that the numerical convergence of cycle expansions for systems such as the 3-disk game of pinball, table 18.2, is very impressive; only three input numbers (the two fixed points $\overline{0}$, $\overline{1}$ and the 2-cycle $\overline{10}$) already yield the escape rate to 4 significant digits! We have omitted an infinity of unstable cycles; so why does approximating the dynamics by a finite number of cycles work so well?

Looking at the cycle expansions simply as sums of unrelated contributions is not specially encouraging: the cycle expansion (18.2) is not absolutely convergent in the sense of Dirichlet series of sect. 18.6, so what one makes of it depends on the way the terms are arranged.

The simplest estimate of the error introduced by approximating smooth flow by periodic orbits is to think of the approximation as a tessalation of a smooth curve by piecewise linear tiles, figure 1.12.

I.1.1 Tessalation of a smooth flow by cycles

One of the early high accuracy computations of π was due to Euler. Euler computed the circumference of the circee of unit radius by inscribing into it a regular polygon with N sides; the error of such computation is proportional to $1 - \cos(2\pi/N) \propto N^{-2}$. In a periodic orbit tessalation of a smooth flow, we cover the phase space by e^{hn} tiles at the nth level of resolution, where h is the topological entropy, the growth rate of the number of tiles. Hence we expect the error in approximating a smooth flow by e^{hn} linear segments to be exponentially small, of order $N^{-2} \propto e^{-2hn}$.

I.1.2 Shadowing and convergence of curvature expansions

We have shown in chapter 13 that if the symbolic dynamics is defined by a finite grammar, a finite number of cycles, let us say the first k terms in the cycle expansion are necessary to correctly count the pieces of the Cantor set generated by the dynamical system.

They are composed of products of non–intersecting loops on the Markov graph, see (13.13). We refer to this set of non–intersecting loops as the *fundamental* cycles of the strange set. It is only after these terms have been included that the cycle expansion is expected to converge smoothly, i.e., only for $n > k$ are the curvatures c_n in (9.2??) a measure of the variation of the quality of a linearized covering of the dynamical Cantor set by the length n cycles, and expected to fall off rapidly with n.

The rate of fall-off of the cycle expansion coefficients can be estimated by observing that for subshifts of finite type the contributions from longer orbits in curvature expansions such as (18.7) can always be grouped into shadowing combinations of pseudo-cycles. For example, a cycle with itinerary $\overline{ab}= s_1 s_2 \cdots s_n$ will appear in combination of form

$$1/\zeta = 1 - \cdots - (t_{ab} - t_a t_b) - \cdots ,$$

with $\overline{ab}$ shadowed by cycle $\overline{a}$ followed by cycle $\overline{b}$, where $a = s_1 s_2 \cdots s_m$, $b = s_{m+1} \cdots s_{n-1} s_n$, and s_k labels the Markov partition $\mathcal{M}_{s_k}$ (10.4) that the trajectory traverses at the kth return. If the two trajectories coincide in the first m symbols, at the mth return to a Poincaré section they can land anywhere in the phase space $\mathcal{M}$

$$\left| f^{T_a}(x_a) - f^{T_{a\ldots}}(x_{a\ldots}) \right| \approx 1 ,$$

where we have assumed that the $\mathcal{M}$ is compact, and that the maximal possible separation across $\mathcal{M}$ is $O(1)$. Here x_a is a point on the $\overline{a}$ cycle of period T_a, and $x_{a\ldots}$ is a nearby point whose trajectory tracks the cycle $\overline{a}$ for the first m Poincaré section returns completed at the time $T_{a\ldots}$. An estimate of the maximal separation of the initial points of the two neighboring trajectories is achieved by Taylor expanding around $x_{a\ldots} = x_{\overline{a}} + \delta x_{a\ldots}$

$$f^{T_a}(x_{\overline{a}}) - f^{T_{a\ldots}}(x_{a\ldots}) \approx \frac{\partial f^{T_a}(x_{\overline{a}})}{\partial x} \cdot \delta x_{a\ldots} = M_a \cdot \delta x_{a\ldots} ,$$

hence the hyperbolicity of the flow forces the initial points of neighboring trajectories that track each other for at least m consecutive symbols to lie exponentially close

$$|\delta x_{a\ldots}| \propto \frac{1}{|\Lambda_a|} .$$

Similarly, for any observable (15.1) integrated along the two nearby trajectories

$$A^{T_{a\dots}}(x_{a\dots}) \approx A^{T_a}(x_{\overline{a}}) + \left.\frac{\partial A^{T_a}}{\partial x}\right|_{x=x_{\overline{a}}} \cdot \delta x_{a\dots}\,,$$

so

$$\left|A^{T_{a\dots}}(x_{a\dots}) - A^{T_a}(x_{\overline{a}})\right| \propto \frac{T_a \mathrm{Const}}{|\Lambda_a|}\,,$$

As the time of return is itself an integral along the trajectory, return times of nearby trajectories are exponentially close

$$|T_{a\dots} - T_a| \propto \frac{T_a \mathrm{Const}}{|\Lambda_a|}\,,$$

and so are the trajectory stabilities

$$\left|A^{T_{a\dots}}(x_{a\dots}) - A^{T_a}(x_{\overline{a}})\right| \propto \frac{T_a \mathrm{Const}}{|\Lambda_a|}\,,$$

Substituting t_{ab} one finds

$$\frac{t_{ab} - t_a t_b}{t_{ab}} = 1 - e^{-s(T_a + T_b - T_{ab})}\left|\frac{\Lambda_a \Lambda_b}{\Lambda_{ab}}\right|\,.$$

Since with increasing m segments of $\overline{ab}$ come closer to $\overline{a}$, the differences in action and the ratio of the eigenvalues converge exponentially with the eigenvalue of the orbit $\overline{a}$,

$$T_a + T_b - T_{ab} \approx \mathrm{Const} \times \Lambda_a^{-j}\,, \quad |\Lambda_a \Lambda_b / \Lambda_{ab}| \approx \exp(-\mathrm{Const}/\Lambda_{ab})$$

Expanding the exponentials one thus finds that this term in the cycle expansion is of the order of

$$t_{a^j b} - t_a t_{a^{j-1} b} \approx \mathrm{Const} \times t_{a^j b} \Lambda_a^{-j}\,. \tag{I.1}$$

Even though the number of terms in a cycle expansion grows exponentially, the shadowing cancellations improve the convergence by an exponential factor compared to trace formulas, and extend the radius of convergence of the periodic orbit sums. Table I.1 shows some examples of such compensations between long cycles and their pseudo-cycle shadows.

n	$t_{ab} - t_a t_b$	$T_{ab} - (T_a + T_b)$	$\log \left[\frac{\Lambda_a \Lambda_b}{\Lambda_{ab}}\right]$	$ab - a \cdot b$
2	-5.23465150784$\times 10^{4}$	4.85802927371$\times 10^{2}$	-6.3$\times 10^{2}$	01-0·1
3	-7.96028600139$\times 10^{6}$	5.21713101432$\times 10^{3}$	-9.8$\times 10^{3}$	001-0·01
4	-1.03326529874$\times 10^{7}$	5.29858199419$\times 10^{4}$	-1.3$\times 10^{3}$	0001-0·001
5	-1.27481522016$\times 10^{9}$	5.35513574697$\times 10^{5}$	-1.6$\times 10^{4}$	00001-0·0001
6	-1.52544704823$\times 10^{11}$	5.40999882625$\times 10^{6}$	-1.8$\times 10^{5}$	000001-0·00001
2	-5.23465150784$\times 10^{4}$	4.85802927371$\times 10^{2}$	-6.3$\times 10^{2}$	01-0·1
3	5.30414752996$\times 10^{6}$	-3.67093656690$\times 10^{3}$	7.7$\times 10^{3}$	011-01·1
4	-5.40934261680$\times 10^{8}$	3.14925761316$\times 10^{4}$	-9.2$\times 10^{4}$	0111-011·1
5	4.99129508833$\times 10^{10}$	-2.67292822795$\times 10^{5}$	1.0$\times 10^{4}$	01111-0111·1
6	-4.39246000586$\times 10^{12}$	2.27087116266$\times 10^{6}$	-1.0$\times 10^{5}$	011111-01111·1

Table I.1: Demonstration of shadowing in curvature combinations of cycle weights of form $t_{ab} - t_a t_b$, the 3-disk fundamental domain cycles at $R : d = 6$, table 26.3. The ratio $\Lambda_a \Lambda_b / \Lambda_{ab}$ is approaching unity exponentially fast.

It is crucial that the curvature expansion is grouped (and truncated) by topologically related cycles and pseudo-cycles; truncations that ignore topology, such as inclusion of all cycles with $T_p < T_{max}$, will contain orbits unmatched by shadowed orbits, and exhibit a mediocre convergence compared with the curvature expansions.

Note that the existence of a pole at $z = 1/c$ implies that the cycle expansions have a finite radius of convergence, and that analytic continuations will be required for extraction of the non-leading zeros of $1/\zeta$. Preferably, one should work with cycle expansions of Selberg products, as discussed in sect. 18.2.2.

I.1.3 No shadowing, poorer convergence

Conversely, if the dynamics is not of a finite subshift type, there is no finite topological polynomial, there are no "curvature" corrections, and the convergence of the cycle expansions will be poor.

I.2 On importance of pruning

If the grammar is not finite and there is no finite topological polynomial, there will be no "curvature" expansions, and the convergence will be poor. That is the generic case, and one strategy for dealing with it is to find a good sequence of approximate but finite grammars; for each approximate grammar cycle expansions yield exponentially accurate eigenvalues, with successive approximate grammars converging toward the desired infinite grammar system.

When the dynamical system's symbolic dynamics does not have a finite grammar, and we are not able to arrange its cycle expansion into curvature combinations (18.7), the series is truncated as in sect. 18.5, by including all pseudo-cycles such that $|\Lambda_{p_1} \cdots \Lambda_{p_k}| \leq |\Lambda_P|$, where P is the most unstable prime cycle included into truncation. The truncation error should then be of order $O(e^{hT_P}T_P/|\Lambda_P|)$, with h the topological entropy, and e^{hT_P} roughly the number of pseudo-cycles of stability $\approx |\Lambda_P|$. In this case the cycle averaging formulas do not converge significantly better than the approximations such as the trace formula (19.18).

Numerical results (see for example the plots of the accuracy of the cycle expansion truncations for the Hénon map in ref. [18.3]) indicate that the truncation error of most averages tracks closely the fluctuations due to the irregular growth in the number of cycles. It is not known whether one can exploit the sum rules such as the mass flow conservation (19.11) to improve the accuracy of dynamical averaging.

I.3 Ma-the-matical caveats

> "Lo duca e io per quel cammino ascoso intrammo a ritornar nel chiaro monde; e sanza cura aver d'alcun riposa
> salimmo sù, el primo e io secondo, tanto ch'i' vidi de le cose belle che porta 'l ciel, per un perutgio tondo."
>
> —Dante

The periodic orbit theory is learned in stages. At first glance, it seems totally impenetrable. After basic exercises are gone through, it seems totally trivial; all that seems to be at stake are elementary manipulations with traces, determinants, derivatives. But if start thinking about you will get a more and more uncomfortable feeling that from the mathematical point of view, this is a perilous enterprise indeed. In chapter 20 we shall explain which parts of this enterprise are really solid; here you give a fortaste of what objections a mathematician might rise.

Birkhoff's 1931 ergodic theorem states that the time average (15.4) exists almost everywhere, and, if the flow is ergodic, it implies that $\langle a(x)\rangle = \langle a\rangle$ is a constant for almost all x. The problem is that the above cycle averaging formulas implicitly rely on ergodic hypothesis: they are strictly correct only if the dynamical system is locally hyperbolic and globally mixing. If one takes a β derivative of both sides

$$\rho_\beta(y)e^{ts(\beta)} = \int_{\mathcal{M}} dx\, \delta(y - f^t(x))e^{\beta\cdot A^t(x)}\rho_\beta(x)\,,$$

and integrates over y

$$\int_{\mathcal{M}} dy\ \frac{\partial}{\partial\beta}\rho_\beta(y)\bigg|_{\beta=0} \ + \ t\,\frac{\partial s}{\partial\beta}\bigg|_{\beta=0}\int_{\mathcal{M}} dy\,\rho_0(y) \ = \\ \int_{\mathcal{M}} dx\, A^t(x)\rho_0(x) + \int_{\mathcal{M}} dx\ \frac{\partial}{\partial\beta}\rho_\beta(x)\bigg|_{\beta=0}\,,$$

one obtains in the long time limit

$$\left.\frac{\partial s}{\partial \beta}\right|_{\beta=0} = \int_{\mathcal{M}} dy\, \rho_0(x)\, \langle a(x) \rangle \ . \tag{I.2}$$

This is the expectation value (15.12) only if the time average (15.4) equals the space average (15.9), $\langle a(x) \rangle = \langle a \rangle$, for all x except a subset $x \in \mathcal{M}$ of zero measure; if the phase space is foliated into non-communicating subspaces $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$ of finite measure such that $f^t(\mathcal{M}_1) \cap \mathcal{M}_2 = \emptyset$ for all t, this fails. In other words, we have tacitly assumed metric indecomposability or transitivity. We have also glossed over the nature of the "phase space" $\mathcal{M}$. For example, if the dynamical system is open, such as the 3-disk game of pinball, $\mathcal{M}$ in the expectation value integral (15.22) is a Cantor set, the closure of the union of all periodic orbits. Alternatively, $\mathcal{M}$ can be considered continuous, but then the measure ρ_0 in (I.2) is highly singular. The beauty of the periodic orbit theory is that instead of using an arbitrary coordinatization of $\mathcal{M}$ it partitions the phase space by the intrinsic topology of the dynamical flow and builds the correct measure from cycle invariants, the stability eigenvalues of periodic orbits.

Were we to restrict the applications of the formalism only to systems which have been rigorously proven to be ergodic, we might as well fold up the shop right now. For example, even for something as simple as the Hénon mapping we do not know whether the asymptotic time attractor is strange or periodic. Physics applications require a more pragmatic attitude. In the cycle expansions approach we construct the invariant set of the given dynamical system as a closure of the union of periodic orbits, and investigate how robust are the averages computed on this set. This turns out to depend very much on the observable being averaged over; dynamical averages exhibit "phase transitions", and the above cycle averaging formulas apply in the "hyperbolic phase" where the average is dominated by exponentially many exponentially small contributions, but fail in a phase dominated by few marginally stable orbits. Here the noise - always present, no matter how weak - helps us by erasing an infinity of small traps that the deterministic dynamics might fall into.

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Still, in spite of all the caveats, periodic orbit theory is a beautiful theory, and the cycle averaging formulas are the most elegant and powerful tool available today for evaluation of dynamical averages for low dimensional chaotic deterministic systems.

I.4 Estimate of the nth cumulant

An immediate consequence of the exponential spacing of the eigenvalues is that the convergence of the Selberg product expansion (D.12) as function of the topological cycle length, $F(z) = \sum_n C_n z^n$, is faster than exponential. Consider a d–dimensional map for which all fundamental matrix eigenvalues are equal: $u_p = \Lambda_{p,1} = \Lambda_{p,2} = \cdots = \Lambda_{p,d}$. The stability eigenvalues are generally not isotropic; however, to obtain qualitative bounds on the spectrum, we replace all stability eigenvalues with the least expanding one. In this case the p cycle contribution to the product (17.9) reduces to

$$
\begin{aligned}
F_p(z) &= \prod_{k_1 \cdots k_d = 0}^{\infty} \left(1 - t_p u_p^{k_1+k_2+\cdots+k_d}\right) \\
&= \prod_{k=0}^{\infty} \left(1 - t_p u_p^k\right)^{m_k}; \qquad m_k = \binom{d-1+k}{d-1} = \frac{(k+d-1)!}{k!(d-1)!} \\
&= \prod_{k=0}^{\infty} \sum_{\ell=0}^{m_k} \binom{m_k}{\ell} \left(-u_p^k t_p\right)^{\ell}
\end{aligned}
\tag{I.3}
$$

In one dimension the expansion can be given in closed form (20.34), and the coefficients C_k in (D.12) are given by

$$
\tau_{p^k} = (-1)^k \frac{u_p^{\frac{k(k-1)}{2}}}{\prod_{j=1}^{k}(1-u_p^j)} t_p^k \,. \tag{I.4}
$$

Hence the coefficients in the $F(z) = \sum_n C_n z^n$ expansion of the spectral determinant (18.11) fall off faster than exponentially, as $|C_n| \approx u^{n(n-1)/2}$. In contrast, the cycle expansions of dynamical zeta functions fall of "only" exponentially; in numerical applications, the difference is dramatic.

In higher dimensions the expansions are not quite as compact. The leading power of u and its coefficient are easily evaluated by use of binomial expansions (I.3) of the $(1+tu^k)^{m_k}$ factors. More precisely, the leading u^n terms in t^k coefficients are of form

$$
\begin{aligned}
\prod_{k=0}^{\infty} (1+tu^k)^{m_k} &= \ldots + u^{m_1+2m_2+\ldots+jm_j}\, t^{1+m_1+m_2+\ldots+m_j} + \ldots \\
&= \ldots + \left(u^{\frac{m_d}{d+1}} t\right)^{\binom{d+m}{m}} + \ldots \approx \ldots + u^{\frac{\sqrt[d]{d!}}{(d-1)!} n^{\frac{d+1}{d}}} t^n + \ldots
\end{aligned}
$$

Hence the coefficients in the $F(z)$ expansion fall off faster than exponentially, as $u^{n^{1+1/d}}$. The Selberg products are entire functions in any dimension, provided that the symbolic dynamics is a finite subshift, and all cycle eigenvalues are sufficiently bounded away from 1.

The case of particular interest in many applications are the 2-d Hamiltonian mappings; their symplectic structure implies that $u_p = \Lambda_{p,1} = 1/\Lambda_{p,2}$, and the Selberg product (17.13) In this case the expansion corresponding to (20.34) is given by (20.35) and the coefficients fall off asymptotically as $C_n \approx u^{n^{3/2}}$.